Path integral representation for Wilson loops and the non-Abelian Stokes theorem

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We discuss the derivation of the path integral representation over gauge degrees of freedom for Wilson loops in $SU(N)$ gauge theory and 4-dimensional Euclidean space-time by using well-known properties of group characters. A discretized form of the path integral is naturally provided by the properties of group characters and does not need any artificial regularization. We show that the path integral over gauge degrees of freedom for Wilson loops derived by Diakonov and Petrov [Phys. Lett. B **224** 131 (1989)] by using a special regularization is erroneous and predicts zero for the Wilson loop. This property is obtained by direct evaluation of path integrals for Wilson loops defined for pure $SU(2)$ gauge fields and $Z(2)$ center vortices with spatial azimuthal symmetry. Further we show that both derivations given by Diakonov and Petrov for their regularized path integral, if done correctly, predict also zero for Wilson loops. Therefore, the application of their path integral representation of Wilson loops cannot give *a new way to check confinement in lattice* as has been declared by Diakonov and Petrov [Phys. Lett. B **242** 425 (1990)]. From the path integral representation which we consider we conclude that no new non-Abelian Stokes theorem can exist for Wilson loops except the old-fashioned one derived by means of the path-ordering procedure.

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I. INTRODUCTION

The hypothesis of quark confinement, bridging the hypothesis of the existence of quarks and the failure of the detection of quarks as isolated objects, is a challenge for QCD. As a criterion of color confinement in QCD, Wilson [1] suggested to consider the average value of an operator

$$W(C)=\frac{1}{N}\,\mathrm{tr}\,\mathcal{P}_C\, e^{i\,g\oint_C dx_\mu\, A_\mu(x)}=\frac{1}{N}\,\mathrm{tr}\,U(C_{xx}), \qquad (1)$$

defined on an closed loop C, where $A_\mu(x)=t^a A^a_\mu(x)$ is a gauge field, t^a $(a=1,\ldots,N^2-1)$ are the generators of the $SU(N)$ gauge group in fundamental representation normalized by the condition $\mathrm{tr}\,(t^a t^b)=\delta^{ab}/2$, g is the gauge coupling constant and $\mathcal{P}_C$ is the operator ordering color matrices along the path C. The trace in Eq. (1) is computed over color indices. The operator

$$U(C_{yx})=\mathcal{P}_{C_{yx}} e^{i\,g\int_{C_{yx}} dz_\mu\, A_\mu(z)}, \qquad (2)$$

makes a parallel transport along the path C_{yx} from x to y. For Wilson loops the contor C defines a closed path C_{xx}. For determinations of the parallel transport operator $U(C_{yx})$ the action of the path-ordering operator $\mathcal{P}_{C_{yx}}$ is defined by the following limiting procedure [2]:

$$\begin{aligned}U(C_{yx})&=\mathcal{P}_{C_{xy}} e^{i\,g\int_{C_{yx}} dz_\mu\, A_\mu(z)}\\&=\lim_{n\to\infty}\prod_{k=1}^{n} U(C_{x_k x_{k-1}})\\&=\lim_{n\to\infty} U(C_{y x_{n-1}})\,\ldots\,U(C_{x_2 x_1})\,U(C_{x_1 x})\\&=\lim_{n\to\infty}\prod_{k=1}^{n} e^{i\,g\,(x_k-x_{k-1})\cdot A(x_{k-1})},\end{aligned} \qquad (3)$$

where $C_{x_k x_{k-1}}$ is an infinitesimal segment of the path C_{yx} with $x_0=x$ and $x_n=y$. The parallel transport operator $U(C_{x_k x_{k-1}})$ for an infinitesimal segment $C_{x_k x_{k-1}}$ is defined by [2]

$$U(C_{x_k x_{k-1}})=e^{i\,g\int_{C_{x_k x_{k-1}}} dz_\mu\, A_\mu(z)}=e^{i\,g\,(x_k-x_{k-1})\cdot A(x_{k-1})}. \qquad (4)$$

In accordance with the definition of the path-ordering procedure (3) the parallel transport operator $U(C_{yx})$ has the property

$$U(C_{yx})=U(C_{yx_1})\,U(C_{x_1 x}), \qquad (5)$$

where x_1 belongs to the path C_{yx}. Under gauge transformations with a gauge function $\Omega(z)$,

$$A_\mu(z)\to A^\Omega_\mu(z)=\Omega(z)A_\mu(z)\Omega^\dagger(z)+\frac{1}{ig}\,\partial_\mu\Omega(z)\Omega^\dagger(z), \qquad (6)$$

the operator $U(C_{yx})$ has a very simple transformation law

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$$U(C_{yx}) \to U^{\Omega}(C_{yx}) = \Omega(y)\, U(C_{yx})\, \Omega^{\dagger}(x). \tag{7}$$

We would like to stress that this equation is valid even if the gauge functions $\Omega(x)$ and $\Omega(y)$ differ significantly for adjacent points x and y.

As has been postulated by Wilson [1] the average value of the Wilson loop $\langle W(C)\rangle$ in the confinement regime should show area-law falloff [1]:

$$\langle W(C)\rangle \sim e^{-\sigma \mathcal{A}}, \tag{8}$$

where σ and $\mathcal{A}$ are the string tension and the minimal area of the loop, respectively. As usually the minimal area is a rectangle of size $L\times T$. In this case the exponent $\sigma\mathcal{A}$ can be represented in the equivalent form $\sigma\mathcal{A} = V(L)\, T$, where $V(L) = \sigma L$ is the interquark potential and L is the relative distance between quark and anti-quark.

The paper is organized as follows. In Sec. II we discuss the path integral representation for Wilson loops by using well-known properties of group characters. The discretized form of this path integral is naturally provided by properties of group characters and does not need any artificial regularization. We derive a closed expression for Wilson loops in irreducible representation j of $SU(2)$. In Sec. III we extend the path integral representation to the gauge group $SU(N)$. As an example, we give an explicit representation for Wilson loops in the fundamental representation of $SU(3)$. In Secs. IV and V we evaluate the path integral for Wilson loops, suggested in Ref. [3], for two specific gauge field configurations (i) a pure gauge field in the fundamental representation of $SU(2)$ and (ii) $Z(2)$ center vortices with spatial azimuthal symmetry, respectively. We show that this path integral representation fails to describe the original Wilson loop for both cases. In Sec. VI we show that the regularized evolution operator in Ref. [3] representing Wilson loops in the form of the path integral over gauge degrees of freedom has been evaluated incorrectly by Diakonov and Petrov. The correct value for the evolution operator is zero. This result agrees with those obtained in Secs. IV and V. In Sec. VII we criticize the removal of the oscillating factor from the evolution operator suggested in Ref. [3] via a shift of energy levels of the axial-symmetric top. We show that such a removal is prohibited. It leads to a change of symmetry of the starting system from $SU(2)$ to $U(2)$. Keeping the oscillating factor one gets a vanishing value of Wilson loops in agreement with our results in Secs. IV, V, and VI. In the Appendix we evaluate the coefficients of the expansion used for the path integrals in Secs. IV and V.

II. PATH INTEGRAL REPRESENTATION FOR WILSON LOOPS

Attempts to derive a path integral representation for Wilson loops (1), where the path ordering operator is replaced by a path integral, have been undertaken in Refs. [3–5]. The path integral representations have been derived for Wilson loops in terms of gauge degrees of freedom (bosonic variables) [3,4] and fermionic degrees of freedom (Grassmann variables) [5]. For the derivation of the quoted path integral representations for Wilson loop different mathematical machineries have been used. Below we discuss the derivation of the path integral representation for Wilson loops in terms of gauge degrees of freedom by using well-known properties of group characters. In this case a discretized form of path integrals is naturally provided by the properties of group characters and the completeness condition of gauge functions. It coincides with the standard discretization of Feynman path integrals [6] and does not need any artificial regularization.

We argue that the path integral representation for Wilson loops suggested by Diakonov and Petrov in Ref. [3] is erroneous. For the derivation of this path integral representation Diakonov and Petrov have used a special regularization drawing an analogy with an axial-symmetric top. The moments of inertia of this top are taken finally to zero. As we show below this path integral amounts to zero for Wilson loops defined for $SU(2)$. Therefore, it is not a surprise that the application of this erroneous path integral representation to the evaluation of the average value of Wilson loops has led to the conclusion that for large loops the area-law falloff is present for color charges taken in any irreducible representation r of $SU(N)$ [7]. This statement has not been supported by numerical simulations within lattice QCD [8]. As has been verified, e.g. in Ref. [8] for $SU(3)$, in the confined phase and at large distances, color charges with non-zero N-ality have string tensions of the corresponding fundamental representation, whereas color charges with zero N-ality are screened by gluons and cannot form a string at large distances. Hence, the results obtained in Ref. [6] cannot give *a new way to check confinement in lattice* as has been declared by Diakonov and Petrov.

For the derivation of Wilson loops in the form of a path integral over gauge degrees of freedom by using well-known properties of group characters it is convenient to represent $W(C)$ in terms of characters of irreducible representations of $SU(N)$ [9–11]

$$W_r(C) = \frac{1}{d_r}\chi[U_r(C_{xx})], \tag{9}$$

where the matrix $U_r(C_{xx})$ realizes an irreducible and d_r-dimensional matrix representation r of the group $SU(N)$ with the character $\chi[U_r(C_{xx})] = \mathrm{tr}[U_r(C_{xx})]$.

In order to introduce the path integral over gauge degrees of freedom we suggest to use

$$\int D\Omega_r \chi[U_r\Omega_r^{\dagger}]\,\chi[\Omega_r V_r] = \frac{1}{d_r}\chi[U_r V_r], \tag{10}$$

where the matrices U_r and V_r belong to the irreducible representation r, and $D\Omega_r$ is the Haar measure normalized to unity $\int D\Omega_r = 1$. The completeness condition for gauge functions Ω_r reads

$$\int D\Omega_r(\Omega_r^\dagger)_{a_1b_1}(\Omega_r)_{a_2b_2}=\frac{1}{d_r}\,\delta_{a_1b_2}\,\delta_{b_1a_2}. \qquad (11)$$

By using the completeness condition it is convenient to represent the Wilson loop in the form of the integral

$$W_r(C)=\frac{1}{d_r}\int D\Omega_r(x)\chi[\Omega_r(x)U_r(C_{xx})\Omega_r^\dagger(x)]. \quad (12)$$

According to Eq. (3) and Eq. (5) the matrix $U_r(C_{xx})$ can be decomposed in

$$U_r(C_{xx})=\lim_{n\to\infty}U_r(C_{xx_{n-1}})U_r(C_{x_{n-1}x_{n-2}})\ldots U_r(C_{x_2x_1})\times U_r(C_{x_1x}). \qquad (13)$$

Substituting Eq. (13) in Eq. (12) and applying $(n-1)$–times Eq. (10) we end up with

$$W_r(C)=\frac{1}{d_r^2}\lim_{n\to\infty}\int\ldots\int D\Omega_r(x_1)\ldots\Omega_r(x_n)\times d_r\chi[\Omega_r(x_n)U_r(C_{x_nx_{n-1}})\Omega_r^\dagger(x_{n-1})]\ldots d_r\chi\times[\Omega_r(x_1)U_r(C_{x_1x_n})\Omega_r^\dagger(x_n)]. \qquad (14)$$

Using relations $\Omega_r(x_k)U_r(C_{x_kx_{k-1}})\Omega_r^\dagger(x_{k-1})=U_r^\Omega(C_{x_kx_{k-1}})$ we get

$$W_r(C)=\frac{1}{d_r^2}\lim_{n\to\infty}\int\ldots\int D\Omega_r(x_1)\ldots D\Omega_r(x_n)\times d_r\chi[U_r^\Omega(C_{x_nx_{n-1}})]\ldots d_r\chi[U_r^\Omega(C_{x_1x_n})]. \qquad (15)$$

The integrations over $\Omega_r(x_k)$ $(k=1,\ldots,n)$ are well defined. These are standard integrations on the compact Lie group $SU(N)$.

We should emphasize that the integrations over $\Omega_r(x_k)$ $(k=1,\ldots,n)$ are not correlated and should be carried out independently.

Since Eq. (11) is the completeness condition for group elements, the discretization of Wilson loops given by Eqs. (14) and (15) reproduces the standard discretization of Feynman path integrals [6] where infinitesimal time steps can be described by a classical motion. Therefore, the discretized expression (15) can be represented formally by

$$W_r(C)=\frac{1}{d_r^2}\int\prod_{x\in C}[d_r\,D\Omega_r(x)]\,\chi[U_r^\Omega(C_{xx})]. \qquad (16)$$

Conversely the evaluation of this path integral corresponds to the discretization given by Eqs. (14) and (15). The measure of the integration over $\Omega_r(x)$ is well defined and normalized to unity

$$\int\prod_{x\in C}D\Omega_r(x)=\lim_{n\to\infty}\int D\Omega_r(x_n)\times\int D\Omega_r(x_{n-1})\ldots\int D\Omega_r(x_1)=1. \qquad (17)$$

Thus, for the determination of the path integral over gauge degrees of freedom (16) we do not need to use any regularization, since the discretization given by Eqs. (14) and (15) are well defined.

We would like to emphasize that Eq. (16) is a continuum analogy of the lattice version of the path integral over gauge degrees of freedom for Wilson loops used in Eq. (2.13) of Ref. [11] for the evaluation of the average value of Wilson loops in connection with $Z(2)$ center vortices.

Now let us proceed to the evaluation of the characters $\chi[U_r^\Omega(C_{x_kx_{k-1}})]$. Due to the infinitesimality of the segments $C_{x_kx_{k-1}}$ we can omit the path ordering operator in the definition of $U_r^\Omega(C_{x_kx_{k-1}})$ [2]. This allows us to evaluate the character $\chi[U_r^\Omega(C_{x_kx_{k-1}})]$ with $U_r^\Omega(C_{x_kx_{k-1}})$ taken in the form [2]

$$U_r^\Omega(C_{x_kx_{k-1}})=\exp ig\int_{C_{x_kx_{k-1}}}dx_\mu A_\mu^\Omega(x). \qquad (18)$$

Of course, the relation given by Eq. (18) is only defined in the sense of a meanvalue over an infinitesimal segment $C_{x_kx_{k-1}}$. Therefore, it can be regarded to some extent as a smoothness condition. Unlike the smoothness condition used by Diakonov and Petrov [3] Eq. (18) does not corrupt the Wilson loop represented by the path integral over the gauge degrees of freedom.

The evaluation of the characters of $U_r^\Omega(C_{x_kx_{k-1}})$ given by Eq. (18) runs as follows. First let us consider the simplest case, the $SU(2)$ gauge group, where we have $r=j=0,1/2,1,\ldots$ and $d_j=2j+1$. The character $\chi[U_j^\Omega(C_{x_kx_{k-1}})]$ is equal to [9,10,12]

$$\chi[U_j^\Omega(C_{x_kx_{k-1}})]=\sum_{m_j=-j}^{j}\langle m_j|U_j^\Omega(C_{x_kx_{k-1}})|m_j\rangle=\sum_{m_j=-j}^{j}e^{i\,m_j\,\Phi[C_{x_kx_{k-1}};A^\Omega]}, \qquad (19)$$

where m_j is the magnetic color quantum number, $|m_j\rangle$ and $m_j\,\Phi[C_{x_kx_{k-1}};A^\Omega]$ are the eigenstates and eigenvalues of the operator

$$\hat{\Phi}[C_{x_kx_{k-1}};A^\Omega]=g\int_{C_{x_kx_{k-1}}}dx_\mu A_\mu^\Omega(x), \qquad (20)$$

i.e. $\hat{\Phi}[C_{x_kx_{k-1}};A^\Omega]\,|m_j\rangle=m_j\,\Phi[C_{x_kx_{k-1}};A^\Omega]\,|m_j\rangle$. The standard procedure for the evaluation of the eigenvalues gives $\Phi[C_{x_kx_{k-1}};A^\Omega]$ in the form

$$\Phi[C_{x_k x_{k-1}};A^\Omega]=g\int_{C_{x_k x_{k-1}}}\sqrt{g_{\mu\nu}[A^\Omega](x)dx_\mu dx_\nu}, \quad (21)$$

where the metric tensor can be given formally by the expression

$$g_{\mu\nu}[A^\Omega](x)=2\,\mathrm{tr}[A^\Omega_\mu A^\Omega_\nu](x). \quad (22)$$

In order to find an explicit expression for the metric tensor we should fix a gauge. As an example let us take the Fock-Schwinger gauge

$$x_\mu A_\mu(x)=0. \quad (23)$$

In this case the gauge field $A_\mu(x)$ can be expressed in terms of the field strength tensor $G_{\mu\nu}(x)$ as follows:

$$A_\mu(x)=\int_0^1 ds\, s\, x_\alpha\, G_{\alpha\mu}(xs). \quad (24)$$

This can be proven by using the obvious relation

$$\begin{aligned}x_\alpha G_{\alpha\mu}(x)&=x_\alpha\partial_\alpha A_\mu(x)-x_\alpha\partial_\mu A_\alpha(x)-ig[x_\alpha A_\alpha(x),A_\mu(x)]\\&=A_\mu(x)+x_\alpha\frac{\partial}{\partial x_\alpha}A_\mu(x), \quad (25)\end{aligned}$$

valid for the Fock-Schwinger gauge $x_\alpha A_\alpha(x)=0$. Replacing $x\to xs$ we can represent the right-hand side (RHS) of Eq. (25) as a total derivative with respect to s

$$sx_\alpha G_{\alpha\mu}(xs)=A_\mu(xs)+x_\alpha\frac{\partial}{\partial x_\alpha}A_\mu(xs)=\frac{d}{ds}[sA_\mu(xs)]. \quad (26)$$

Integrating out $s\in[0,1]$ we arrive at Eq. (24).

Using Eq. (24) we obtain the metric tensor $g_{\mu\nu}[A^\Omega](x)$ in the form

$$\begin{aligned}g_{\mu\nu}[A^\Omega](x)&=2x_\alpha x_\beta\int_0^1\int_0^1 ds ds' ss'\,\mathrm{tr}[G^\Omega_{\alpha\mu}(xs)G^\Omega_{\beta\nu}(xs')]\\&=2x_\alpha x_\beta\int_0^1\int_0^1 ds ds' ss'\,\mathrm{tr}[\Omega(xs)G_{\alpha\mu}(xs)\\&\quad\times\Omega^\dagger(xs)\Omega(xs')G_{\beta\nu}(xs')\Omega^\dagger(xs')]. \quad (27)\end{aligned}$$

For the derivation of Eq. (27) we define the operator $\Phi[C_{x_k x_{k-1}};A^\Omega]$ of Eq. (20) following the definition of the phase of the parallel transport operator $U(C_{x_k x_{k-1}})$ given by Eq. (4) [2]

$$\begin{aligned}\hat{\Phi}[C_{x_k x_{k-1}};A^\Omega]&=g\int_{C_{x_k x_{k-1}}}dx_\mu A^\Omega_\mu(x)\\&=(x_k-x_{k-1})_\mu A^\Omega_\mu(x_{k-1})\\&=(x_k-x_{k-1})_\mu\int_0^1 ds s\, x^\alpha_{k-1}\, G^\Omega_{\alpha\mu}(x_{k-1}s). \quad (28)\end{aligned}$$

The parameter s is to some extent an order parameter distinguishing the gauge functions $\Omega(x_k)$ and $\Omega(x_{k-1})$ entering the relation $\Omega(x_k)U(C_{x_k x_{k-1}})\Omega^\dagger(x_{k-1})=U^\Omega(C_{x_k x_{k-1}})$.

Substituting Eq. (19) in Eq. (15) we arrive at the expression for Wilson loops defined for $SU(2)$

$$\begin{aligned}W_j(C)=&\frac{1}{(2j+1)^2}\lim_{n\to\infty}\int D\Omega_j(x_n)\,(2j+1)\sum_{m_j^{(n)}=-j}^{j}e^{i\,g\,m_j^{(n)}\int_{C_{x_1x_n}}\sqrt{g_{\mu\nu}[A^\Omega](x)\,dx_\mu dx_\nu}}\\&\times\int D\Omega_j(x_{n-1})\,(2j+1)\sum_{m_j^{(n-1)}=-j}^{j}e^{i\,g\,m_j^{(n-1)}\int_{C_{x_nx_{n-1}}}\sqrt{g_{\mu\nu}[A^\Omega](x)\,dx_\mu dx_\nu}}\\&\vdots\\&\times\int D\Omega_j(x_1)\,(2j+1)\sum_{m_j^{(1)}=-j}^{j}e^{i\,g\,m_j^{(1)}\int_{C_{x_2x_1}}\sqrt{g_{\mu\nu}[A^\Omega](x)\,dx_\mu dx_\nu}}. \quad (29)\end{aligned}$$

The magnetic quantum number $m_j^{(k)}$ $(k=1,\ldots,n)$ belongs to the infinitesimal segment $C_{x_{k+1}x_k}$, where $C_{x_{n+1}x_n}=C_{x_1x_n}$.

In compact form Eq. (29) can be written as a path integral over gauge functions

$$W_j(C)=\frac{1}{(2j+1)^2}\int\prod_{x\in C}D\Omega_j(x)\sum_{\{m_j(x)\}}(2j+1)\,e^{ig\oint_C m_j(x)}\sqrt{g_{\mu\nu}[A^\Omega](x)\,dx_\mu dx_\nu}. \quad (30)$$

The integrals along the infinitesimal segments $C_{x_k x_{k-1}}$ we determine as [2]

$$\int_{C_{x_kx_{k-1}}} m_j(x)\ \sqrt{g_{\mu\nu}[A^{\Omega}](x)\,dx_\mu dx_\nu}=m_j(x_{k-1})\ \sqrt{g_{\mu\nu}[A^{\Omega}](x_{k-1})\,\Delta x_\mu\,\Delta x_\nu}=m_j^{(k-1)}\sqrt{g_{\mu\nu}[A^{\Omega}](x_{k-1})\,\Delta x_\mu\,\Delta x_\nu}, \quad (31)$$

where $\Delta x = x_k - x_{k-1}$.

Comparing the path integral (30) with that suggested in Eq. (23) of Ref. [3] one finds rather strong disagreement. First, this concerns the contribution of different states m_j of the representation j. In the case of the path integral (30) there is a summation over all values of the magnetic color quantum number m_j, whereas the representation of Ref. [3] contains only one term with $m_j = j$. Second, Ref. [3] claims that in the integrand of their path integral the exponent should depend only on the gauge field projected onto the third axis in color space. However, this is only possible if the gauge functions are slowly varying with x, i.e., $\Omega(x_k)\Omega^\dagger(x_{k-1}) \simeq 1$. In this case the parallel transport operator $U^{\Omega}(C_{x_kx_{k-1}})$ would read [13]

$$U^{\Omega}(C_{x_kx_{k-1}}) = \exp ig\int_{C_{x_kx_{k-1}}} dx_\mu A^{\Omega}_\mu(x) = 1 + i\,g\,(x_i - x_{i-1})\cdot A^{\Omega}(x_{i-1}), \quad (32)$$

and the evaluation of the character $\chi[U^{\Omega}_j(C_{x_kx_{k-1}}]$ would run as follows:

$$\begin{aligned}\langle m_j|[U^{\Omega}_j(C_{x_kx_{k-1}})]|m_j\rangle &= 1 + (t^a_j)_{m_jm_j}\, i\,g\,(x_k - x_{k-1})\cdot[A^{\Omega}(x_{k-1})]^{(a)}\\ &= 1 + m_j\, i\,g\,(x_k - x_{k-1})\cdot[A^{\Omega}(x_{k-1})]^{(3)}\\ &= e^{i\,g\int_{C_{x_kx_{k-1}}} dx_\mu\, m_j(x)[A^{\Omega}_\mu(x)]^{(3)}}, \end{aligned} \quad (33)$$

where we have used the matrix elements of the generators of $SU(2)$, i.e. $(t^a_j)_{m_jm_j} = m_j\,\delta^{a3}$. More generally the exponent on the RHS of Eq. (33) can be written as

$$\int_{C_{x_kx_{k-1}}} dx_\mu\, m_j(x)\,[A^{\Omega}_\mu(x)]^{(3)} = 2\int_{C_{x_kx_{k-1}}} dx_\mu\, m_j(x)\,\mathrm{tr}[t^3_j A^{\Omega}_\mu(x)]. \quad (34)$$

This gives the path integral representation for Wilson loops defined for $SU(2)$ in the following form:

$$W_j(C) = \frac{1}{(2j+1)^2}\int \prod_{x\in C} D\Omega_j(x) \sum_{\{m_j(x)\}} (2j+1) \times e^{2ig\oint_C dx_\mu\, m_j(x)\,\mathrm{tr}[t^3_j A^{\Omega}_\mu(x)]}. \quad (35)$$

The exponent contains the gauge field projected onto the third axis in color space $\mathrm{tr}[t^3_j A^{\Omega}_\mu(x)]$. Nevertheless, Eq. (35) differs from Eq. (23) of Ref. [3] by a summation over all values of the color magnetic quantum number m_j of the given irreducible representation j.

The repeated application of Eq. (10) induces that the integrations over the gauge function at x_k are completely independent of the integrations at $x_{k\pm1}$. There is no mechanism which leads to gauge functions smoothly varying with $x_k\,(k=1,\ldots,n)$. In this sense the situation is opposite to the quantum mechanical path integral. In quantum mechanics the integration over all paths is restricted by the kinetic term of the Lagrange function. In the semiclassical limit $\hbar\to 0$ due to the kinetic term the fluctuations of all trajectories are shrunk to zero around a classical trajectory. However, in the case of the integration over gauge functions for the path integral representation of the Wilson loop, there is neither a suppression factor nor a semiclassical limit like $\hbar\to 0$. The key point of the application of Eq. (10) and, therefore, the path integral representation for Wilson loops is that all integrations over $\Omega(x_k)\,(k=1,\ldots,n)$ are completely independent and can differ substantially even if the points, where the gauge functions $\Omega(x_k)$ and $\Omega(x_{k-1})$ are defined, are infinitesimally close to each other.

For the derivation of Eq. (23) of Ref. [3] Diakonov and Petrov have used at an intermediate step a regularization drawing an analogy with an axial-symmetric top with moments of inertia $I_\perp$ and $I_\parallel$. Within this regularization the evolution operator representing Wilson loops has been replaced by a path integral over dynamical variables of this axial-symmetric top which correspond to gauge degrees of freedom of the non-Abelian gauge field. The regularized expression of the evolution operator has been obtained in the limit $I_\perp, I_\parallel \to 0$. The moments of inertia have been used as parameters like $\hbar\to 0$. Unfortunately, as we show in Sec. VI the limit $I_\perp, I_\parallel\to 0$ has been evaluated incorrectly.

III. THE $SU(N)$ EXTENSION

The extension of the path integral representation given in Eq. (32) to $SU(N)$ is rather straightforward and reduces to the evaluation of the character of the matrix $U^{\Omega}_r(C_{x_kx_{k-1}})$ in the irreducible representation r of $SU(N)$. The character can be given by [12]

$$\chi[U^{\Omega}_r(C_{x_kx_{k-1}})] = \mathrm{tr}\left(e^{i\sum_{l=1}^{N-1} H_l\Phi_l[C_{x_kx_{k-1}};A^{\Omega}]}\right) = \sum_{\vec{m}_r} \gamma_{\vec{m}_r}\, e^{i\,\vec{m}_r\cdot\vec{\Phi}[C_{x_kx_{k-1}};A^{\Omega}]}, \quad (36)$$

where $H_l\,(l=1,\ldots,N-1)$ are diagonal $d_r\times d_r$ traceless matrices realizing the representation of the Cartan subalgebra, i.e. $[H_l, H_{l'}]=0$, of the generators of the $SU(N)$ [12]. The sum runs over all the weights $\vec{m}_r = (m_{r\,1},\ldots,m_{r\,N-1})$

of the irreducible representation r and $\gamma_{\vec{m}_r}$ is the multiplicity of the weight $\vec{m}_r$ and $\Sigma_{\vec{m}_r}\gamma_{\vec{m}_r}=d_r$. The components of the vector $\vec{\Phi}[C_{x_k x_{k-1}};A^{\Omega}]$ are defined by

$$\Phi_l[C_{x_k x_{k-1}};A^{\Omega}]=g\int_{C_{x_k x_{k-1}}}\varphi_l[\omega(x)], \tag{37}$$

where we have introduced the notation $\omega(x)=t^a\omega^a(x)=dz\cdot A^{\Omega}(x)$. The functions $\varphi_l[\omega(x)]$ are proportional to the roots of the equation $\det[\omega(x)-\lambda]=0$.

The path integral representation of Wilson loops defined for the irreducible representation r of $SU(N)$ reads

$$W_r(C)=\frac{1}{d_r^2}\int\prod_{x\in C}D\Omega_r(x)\sum_{\{\vec{m}_r(x)\}}d_r\,\gamma_{\vec{m}_r(x)}\times e^{i\,g\oint_C\vec{m}_r(x)\cdot\vec{\varphi}[\omega(x)]}. \tag{38}$$

Let us consider in more details the path integral representation of Wilson loops defined for the fundamental representation $\underline{3}$ of $SU(3)$. The character $\chi_{\underline{3}}[U^{\Omega}_{\underline{3}}(C)]$ is defined as

$$\begin{aligned}\chi_{\underline{3}}[U^{\Omega}_{\underline{3}}(C)]&=\mathrm{tr}(e^{iH_1\Phi_1[C;A^{\Omega}]+iH_2\Phi_2[C;A^{\Omega}]})\\&=e^{-i\Phi_2[C;A^{\Omega}]/3}+e^{i\Phi_1[C;A^{\Omega}]/2\sqrt{3}}\,e^{i\Phi_2[C;A^{\Omega}]/6}\\&\quad+e^{i\Phi_1[C;A^{\Omega}]/2\sqrt{3}}\,e^{i\Phi_2[C;A^{\Omega}]/6},\end{aligned} \tag{39}$$

where $H_1=t^3/\sqrt{3}$ and $H_2=t^8/\sqrt{3}$ [12]. For the representation $\underline{3}$ of $SU(3)$ the equation $\det[\omega-\lambda]=0$ takes the form

$$\lambda^3-\lambda\,\frac{1}{2}\,\mathrm{tr}\,\omega^2(x)-\det\omega(x)=0. \tag{40}$$

The roots of Eq. (40) read

$$\begin{aligned}\lambda^{(1)}=&-\frac{1}{\sqrt{6}}\sqrt{\mathrm{tr}\,\omega^2(x)}\cos\left(\frac{1}{3}\arccos\sqrt{2\det\left[1+12\frac{t^a\mathrm{tr}(t^a\omega^2(x))}{\mathrm{tr}\,\omega^2(x)}\right]}\right)\\&-\frac{1}{\sqrt{2}}\sqrt{\mathrm{tr}\,\omega^2(x)}\sin\left(\frac{1}{3}\arccos\sqrt{2\det\left[1+12\frac{t^a\mathrm{tr}(t^a\omega^2(x))}{\mathrm{tr}\,\omega^2(x)}\right]}\right),\\ \lambda^{(2)}=&-\frac{1}{\sqrt{6}}\sqrt{\mathrm{tr}\,\omega^2(x)}\cos\left(\frac{1}{3}\arccos\sqrt{2\det\left[1+12\frac{t^a\mathrm{tr}(t^a\omega^2(x))}{\mathrm{tr}\,\omega^2(x)}\right]}\right)\\&+\frac{1}{\sqrt{2}}\sqrt{\mathrm{tr}\,\omega^2(x)}\sin\left(\frac{1}{3}\arccos\sqrt{2\det\left[1+12\frac{t^a\mathrm{tr}(t^a\omega^2(x))}{\mathrm{tr}\,\omega^2(x)}\right]}\right),\\ \lambda^{(3)}=&\sqrt{\frac{2}{3}}\sqrt{\mathrm{tr}\,\omega^2(x)}\cos\left(\frac{1}{3}\arccos\sqrt{2\det\left[1+12\frac{t^a\mathrm{tr}(t^a\omega^2(x))}{\mathrm{tr}\,\omega^2(x)}\right]}\right).\end{aligned} \tag{41}$$

In terms of the roots $\lambda^{(i)}$ $(i=1,2,3)$ the phases $\Phi_{1,2}[C;A^{\Omega}]$ are defined as

$$\begin{aligned}\Phi_1[C;A^{\Omega}]&=-g\sqrt{6}\oint_C\sqrt{\mathrm{tr}\,\omega^2(x)}\sin\left(\frac{1}{3}\arccos\sqrt{2\det\left[1+12\frac{t^a\mathrm{tr}(t^a\omega^2(x))}{\mathrm{tr}\,\omega^2(x)}\right]}\right),\\ \Phi_2[C;A^{\Omega}]&=-g\sqrt{6}\oint_C\sqrt{\mathrm{tr}\,\omega^2(x)}\cos\left(\frac{1}{3}\arccos\sqrt{2\det\left[1+12\frac{t^a\mathrm{tr}(t^a\omega^2(x))}{\mathrm{tr}\,\omega^2(x)}\right]}\right),\end{aligned} \tag{42}$$

where $\mathrm{tr}\,\omega^2(x)=\frac{1}{2}\,g_{\mu\nu}[A^{\Omega}](x)\,dx_{\mu}dx_{\nu}$. Thus, in the fundamental representation $\underline{3}$ the path integral representation for Wilson loops reads

$$W_{\underline{3}}(C)=\frac{1}{9}\int\prod_{x\in C}[D\Omega_{\underline{3}}(x)\times 3]\,(e^{i\Phi_1[C;A^{\Omega}]/2\sqrt{3}}e^{i\Phi_2[C;A^{\Omega}]/6}+e^{-i\Phi_1[C;A^{\Omega}]/2\sqrt{3}}\,e^{i\Phi_2[C;A^{\Omega}]/6}+e^{-i\Phi_2[C;A^{\Omega}]/3}), \tag{43}$$

where the phases $\Phi_{1,2}[C;A^{\Omega}]$ are given by Eq. (42).

IV. WILSON LOOP FOR PURE GAUGE FIELD

As has been pointed out in Ref. [3] the path integral over gauge degrees of freedom representing Wilson loops *is not of the Feynman type, therefore, it depends explicitly on how one ''understands'' it, i.e. how it is discretized and regularized*. We would like to emphasize that the *regularization procedure* applied in Ref. [3] has led to an expression for Wilson loops which supports the hypothesis of maximal Abelian projection [14]. According to this hypothesis only Abelian degrees of freedom of non-Abelian gauge fields are responsible for confinement. This is to full extent a dynamical hypothesis. It is quite obvious that such a dynamical hypothesis cannot be derived only by means of a regularization procedure.

In order to show that the problem touched in this paper is not of marginal interest and to check if path integral expressions that look differently superficially could actually compute the same number we evaluate below explicitly the path integrals representing Wilson loop for a pure $SU(2)$ gauge field. As has been stated in Ref. [3] for Wilson loops C a gauge field *along a given curve can be always written as a ''pure gauge''* and the derivation of the path integral representation for Wilson loops can be provided for the gauge field taken *without loss of generality in the ''pure gauge'' form*. We would like to show that for the pure $SU(2)$ gauge field the path integral representation for Wilson loops suggested in Ref. [3] fails for a correct description of Wilson loops. Since a pure gauge field is equivalent to a zero gauge field Wilson loops should be unity.

Of course, any correct path integral representation for Wilson loops should lead to the same result. The evaluation of Wilson loops within the path integral representation Eq. (16) is rather trivial and transparent. Indeed, we have not corrupted the starting expression for Wilson loops (9) by any artificial regularization. Thereby, the general formula (16) evaluated through the discretization given by Eqs. (15) and (14) is completely identical to the original expression (9). The former gives a unit value for Wilson loops defined for an arbitrary contour C and an irreducible representation J of $SU(2)$: $W_J(C)=1$.

Let us focus now on the path integral representation suggested in Ref. [3]:

$$W_J(C)=\int\prod_{x\in C}D\Omega(x)\,e^{2iJg\oint_C dx_\mu\,\mathrm{tr}[t^3A^{\Omega}_\mu(x)]},\qquad(44)$$

where all matrices are taken in the irreducible representation J. Following the discretization suggested in Ref. [3] we arrive at the expression

$$W_J(C)=\lim_{n\to\infty}\prod_{k=1}^{n}\int D\Omega(x_k)\,e^{2iJg\int_{C_{x_{k+1}x_k}}dx_\mu\,\mathrm{tr}[t^3A^{\Omega}_\mu(x)]}.\qquad(45)$$

Setting $A_\mu(x)=\partial_\mu U(x)U^\dagger(x)/ig$ we get

$$A^{\Omega}_\mu(x)=\frac{1}{ig}\partial_\mu(\Omega(x)U(x))(\Omega(x)U(x))^\dagger.\qquad(46)$$

By a gauge transformation $\Omega(x)U(x)\to\Omega(x)$ we reduce Eq. (44) to the form

$$W_J(C)=\int\prod_{x\in C}D\Omega(x)\,e^{2J\oint_C dx_\mu\,\mathrm{tr}[t^3\partial_\mu\Omega(x)\Omega^\dagger(x)]}.\qquad(47)$$

For simplicity we consider Wilson loops in the fundamental representation of $SU(2)$, $W_{1/2}(C)$. The result can be generalized to any irreducible representation J.

For the evaluation of the path integral Eq. (47) it is convenient to use a standard s parametrization of Wilson loops C [2]: $x_\mu\to x_\mu(s)$, with $s\in[0,1]$ and $x_\mu(0)=x_\mu(1)=x_\mu$.

The Wilson loop (47) reads in the s parametrization

$$W_{1/2}(C)=\int\prod_{0\leqslant s\leqslant 1}D\Omega(s)\exp\int_0^1 ds\,\mathrm{tr}\left[t^3\frac{d\Omega(s)}{ds}\Omega^\dagger(s)\right].\qquad(48)$$

The discretized form of the path integral (48) is given by

$$\begin{aligned}W_J(C)&=\lim_{n\to\infty}\int\prod_{k=1}^{n}D\Omega_k\exp\Delta s_{k+1,k}\,\mathrm{tr}\left[t^3\frac{\Omega_{k+1}-\Omega_k}{\Delta s_{k+1,k}}\Omega_k^\dagger\right]\\&=\lim_{n\to\infty}\int\prod_{k=1}^{n}D\Omega_k\,e^{\mathrm{tr}[t^3\Omega_{k+1}\Omega_k^\dagger]}\\&=\lim_{n\to\infty}\int\dots\int D\Omega_nD\Omega_{n-1}D\Omega_{n-2}\dots D\Omega_1\times e^{\mathrm{tr}[t^3\Omega_n\Omega^\dagger_{n-1}]}\,e^{2J\mathrm{tr}[t^3\Omega_{n-1}\Omega^\dagger_{n-2}]}\dots e^{\mathrm{tr}[t^3\Omega_2\Omega_1^\dagger]}\,e^{\mathrm{tr}[t^3\Omega_1\Omega_n^\dagger]},\end{aligned}\qquad(49)$$

where $\Omega_{n+1}=\Omega_1$.

For the subsequent integration over Ω_k we suggest to use a formula of Ref. [15] modified for our case

$$\int D\Omega\,e^{z\,\mathrm{tr}[t^3A\Omega^\dagger+Bt^3\Omega]}=\sum_j\frac{a_j^2(z)}{2j+1}\chi_j[(t^3)^2AB],\qquad(50)$$

where the coefficients $a_j(z)$ are defined by the expansion [15]

$$e^{z\,\mathrm{tr}[t^3U]}=\sum_j a_j(z)\,\chi_j[t^3U].\qquad(51)$$

In the particular case $z=2J$ and for the fundamental representation $J=1/2$ we have $z=1$. The trace $\mathrm{tr}[t^3U]$ in the exponent of the LHS of Eq. (51) should be evaluated for the fundamental representation of $SU(2)$. By virtue of the orthogonality relation for characters [9,10,15]

$$\int DU\,\chi_j[AU^\dagger]\,\chi_{j'}[UB]=\frac{\delta_{jj'}}{2j+1}\chi_j[AB], \tag{52}$$

where DU is the Haar measure for the $SU(2)$ group, the coefficients $a_j(z)$ for $j\neq 0$ can be determined by [15]

$$a_j(z)=\frac{3}{j(j+1)}\int DU\,\chi_j[t^3U^\dagger]\,e^{z\,\mathrm{tr}[t^3U]}. \tag{53}$$

We have used here that $\chi_j[(t^3)^2]=j(j+1)(2j+1)/3$. The coefficient $a_0(z)$ is defined by

$$a_0(z)=\int DU\,e^{z\,\mathrm{tr}[t^3U]}. \tag{54}$$

The coefficients $a_j(z)$ obey a completeness condition. For its derivation we notice that $\mathrm{tr}[t^3U^\dagger]=-\mathrm{tr}[t^3U]$ which can be easily seen from the standard parametrization of the matrix U in terms of an angle φ and a unit vector $\vec{n}$ [see Eq. (3.96) of Ref. [10]]

$$\begin{aligned} U&=e^{+i\varphi\,\vec{n}\cdot\vec{\tau}/2}=\cos\frac{\varphi}{2}+i(\vec{n}\cdot\vec{\tau})\sin\frac{\varphi}{2},\\ U^\dagger&=e^{-i\varphi\,\vec{n}\cdot\vec{\tau}/2}=\cos\frac{\varphi}{2}-i(\vec{n}\cdot\vec{\tau})\sin\frac{\varphi}{2}, \end{aligned} \tag{55}$$

The expansion of $e^{z\,\mathrm{tr}[t^3U^\dagger]}$ we represent as follows:

$$e^{z\,\mathrm{tr}[t^3U^\dagger]}=\sum_j b_j(z)\,\chi_j[t^3U^\dagger]. \tag{56}$$

Let us show that $b_j(z)=a_j(z)$. Using the orthogonality relation (52) we obtain

$$\begin{aligned} b_j(z)&=\frac{3}{j(j+1)}\int DU\,\chi_j[t^3U]\,e^{z\,\mathrm{tr}[t^3U^\dagger]},\quad j\neq 0,\\ b_0(z)&=\int DU\,e^{z\,\mathrm{tr}[t^3U^\dagger]}. \end{aligned} \tag{57}$$

Then, making the change $U^\dagger\to U$ we get $b_j(z)=a_j(z)$ by virtue of the invariance of the Haar measure $DU^\dagger=DU$.

Thus, taking the product of the expansions (51) and (56) with $b_j(z)=a_j(z)$ and integrating over U we get

$$\begin{aligned} &\int DU\,e^{z\,\mathrm{tr}[t^3U]}\,e^{z\,\mathrm{tr}[t^3U^\dagger]}\\ &=\sum_j\sum_{j'} a_j(z)\,a_{j'}(z)\int DU\,\chi_j[t^3U^\dagger]\,\chi_{j'}[t^3U]\\ &=\sum_j\frac{a_j^2(z)}{2j+1}\chi_j[(t^3)^2]=a_0^2(z)+\sum_{j>0}\frac{1}{3}j(j+1)\,a_j^2(z). \end{aligned} \tag{58}$$

The LHS of Eq. (58) is equal to unity due to the relation $\mathrm{tr}[t^3U^\dagger]=-\mathrm{tr}[t^3U]$ and the normalization of the Haar measure $\int DU=1$. Therefore, the completeness condition for the coefficients $a_j(z)$ reads

$$a_0^2(z)+\sum_{j>0}\frac{1}{3}j(j+1)\,a_j^2(z)=1. \tag{59}$$

The coefficient $a_0(z)$ we evaluate below. The evaluation of coefficients $a_j(z)$ for an arbitrary j is given in the Appendix. For the evaluation of $a_0(z)$ one can use, for example, the standard parametrization (55) and the definition of the Haar measure DU [see Eq. (3.97) of Ref. [10]]

$$DU=\frac{1}{4\pi^2}d\Omega_{\vec{n}}\,d\varphi\,\sin^2\frac{\varphi}{2}, \tag{60}$$

where $d\Omega_{\vec{n}}$ is the uniform measure on the unit sphere S^2 [9]. As a result for $a_0(z)$ we obtain

$$a_0(z)=\int DU\,e^{z\,\mathrm{tr}[t^3U]}=2J_1(z)/z, \tag{61}$$

where $J_1(z)$ is a Bessel function [16]. In the particular case, $z=1$, we get $a_0(1)=a_0=2J_1(1)=0.88$ [16].

For the integration over Ω_k we suppose, first, that n is an even number. Then, integrating over Ω_{n-1}, Ω_{n-3}, $\ldots$, Ω_1 we obtain

$$\begin{aligned} W_{1/2}(C)=\lim_{n\to\infty}\int\ldots\int D\Omega_n D\Omega_{n-2}D\Omega_{n-4}\ldots D\Omega_2\sum_{j_{n-1}}(2j_{n-1}+1)\left[\frac{a_{j_{n-1}}}{2j_{n-1}+1}\right]^2\chi_{j_{n-1}}[(t^3)^2\Omega_n\Omega_{n-2}^\dagger]\\ \times\sum_{j_{n-3}}(2j_{n-3}+1)\left[\frac{a_{j_{n-3}}}{2j_{n-3}+1}\right]^2\chi_{j_{n-3}}[(t^3)^2\Omega_{n-2}\Omega_{n-4}^\dagger]\ldots\sum_{j_4}(2j_4+1)\left[\frac{a_{j_4}}{2j_4+1}\right]^2\chi_{j_4}[(t^3)^2\Omega_4\Omega_2^\dagger]\\ \times\sum_{j_2}(2j_2+1)\left[\frac{a_{j_2}}{2j_2+1}\right]^2\chi_{j_2}[(t^3)^2\Omega_2\Omega_n^\dagger], \end{aligned} \tag{62}$$

where we have denoted $a_j(1)=a_j$.

After the integration over Ω_n, Ω_{n-2}, $\ldots$, Ω_2 we arrive at the expression

$$W_{1/2}(C)=\lim_{n\to\infty}\sum_j (2j+1)\left[\frac{a_j}{2j+1}\right]^n \chi_j[(t^3)^n]$$
$$=\lim_{n\to\infty}\left(a_0^n+\sum_{j>0}(2j+1)\left[\frac{a_j}{2j+1}\right]^n\chi_j[(t^3)^n]\right)$$
$$=\lim_{n\to\infty}\sum_{j>0}(2j+1)\left[\frac{a_j}{2j+1}\right]^n\chi_j[(t^3)^n], \qquad (63)$$

where we have used that $\lim_{n\to\infty}a_0^n=0$ by virtue of the relation $a_0=2J_1(1)=0.88<1$ given by Eq. (61).

The expression (63) is valid too if n is an odd number. However, in this case $\chi_j[(t^3)^n]=0$ and we get immediately $W_{1/2}(C)=0$.

In order to estimate $\chi_j[(t^3)^n]$ for n an even number and at $n\to\infty$ we suggest to apply the following procedure:

$$\chi_j[(t^3)^n]=\sum_{m=-j}^{j} m^n=\frac{\Gamma(n+1)}{2\pi i}\int_{-i\infty}^{i\infty}\sum_{m=-j}^{j}e^{ms}\frac{ds}{s^{n+1}}$$
$$=\frac{\Gamma(n+1)}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{sh}(2j+1)\frac{s}{2}}{\mathrm{sh}\frac{s}{2}}\frac{ds}{s^{n+1}}. \qquad (64)$$

As $n\to\infty$, we can evaluate the integral over s by using the saddle-point approach and get $\chi_j[(t^3)^n]\simeq j^n$.

The Wilson loop is then defined by

$$W_{1/2}(C)=\lim_{n\to\infty}\sum_{j>0}(2j+1)\left[\frac{j\,a_j}{2j+1}\right]^n$$
$$=\lim_{n\to\infty}\sum_{j>0}(2j+1)\exp\left[n\,ln\left|\frac{j\,a_j}{2j+1}\right|\right]. \qquad (65)$$

By using the completeness condition for the coefficients a_j given by Eq. (59) we obtain the constraint

$$\left|\frac{j\,a_j}{2j+1}\right|<\sqrt{3(1-a_0^2)}\sqrt{\frac{j}{j+1}}\frac{1}{2\ j+1}<\sqrt{\frac{j}{j+1}}\frac{1}{2j+1}<1. \qquad (66)$$

This proves that the Wilson loop $W_{1/2}(C)$ vanishes in the limit $n\to\infty$, $W_{1/2}(C)=0$.

Thus, the Wilson loop $W_{1/2}(C)$ for an arbitrary contour C and a pure gauge field represented by the path integral derived in Ref. [3] vanishes, instead of being equal to unity, $W_{1/2}(C)=1$. This shows that the path integral representation suggested in Ref. [3] fails for the correct description of Wilson loops.

V. WILSON LOOP FOR $Z(2)$ CENTER VORTICES

In this section we evaluate explicitly the path integral (44) for Wilson loops pierced by a $Z(2)$ center vortex with spatial azimuthal symmetry. Some problems of $Z(2)$ center vortices with spatial azimuthal symmetry have been analyzed by Diakonov in his recent publication [17] for the gauge group $SU(2)$. In this system the main dynamical variable is the azimuthal component of the non-Abelian gauge field $A_\phi^a(\rho)$ $(a=1,2,3)$ depending only on ρ, the radius in the transversal plane. For a circular Wilson loop in the irreducible representation J one gets

$$W_J(\rho)=\frac{1}{2J+1}\sum_{m=-J}^{J}e^{i2\pi m\mu(\rho)}$$
$$=\frac{1}{2J+1}\frac{\sin[(2J+1)\pi\mu(\rho)]}{\sin[\pi\mu(\rho)]}, \qquad (67)$$

where $\mu(\rho)=\rho\sqrt{A_\phi^a(\rho)A_\phi^a(\rho)}$. The gauge coupling constant g is included in the definition of the gauge field. For Wilson loops in the fundamental representation $J=1/2$ we have

$$W_{1/2}(\rho)=\cos[\pi\mu(\rho)]. \qquad (68)$$

In the case of $Z(2)$ center vortices with spatial azimuthal symmetry and for the fundamental representation of $SU(2)$ Eq. (45) takes the form

$$W_{1/2}(\rho)=\lim_{n\to\infty}\prod_{k=1}^{n}\int D\Omega_k\, e^{\mathrm{tr}[t^3(i2\pi\rho/n)\Omega_{k+1}A_\phi(\rho)\Omega_k^\dagger+t^3\Omega_{k+1}\Omega_k^\dagger]}$$
$$=\lim_{n\to\infty}\int\ldots\int D\Omega_n D\Omega_{n-1}D\Omega_{n-2}\ldots D\Omega_1 e^{\mathrm{tr}[t^3(i2\pi\rho/n)\Omega_n A_\phi(\rho)\Omega_{n-1}^\dagger+t^3\Omega_n\Omega_{n-1}^\dagger]}$$
$$\times e^{\mathrm{tr}[t^3(i2\pi\rho/n)\Omega_{n-1}A_\phi(\rho)\Omega_{n-2}^\dagger+t^3\Omega_{n-1}\Omega_{n-2}^\dagger]}\ldots e^{\mathrm{tr}[t^3(i2\pi\rho/n)\Omega_2 A_\phi(\rho)\Omega_1^\dagger+t^3\Omega_2\Omega_1^\dagger]}$$
$$\times e^{\mathrm{tr}[t^3(i2\pi\rho/n)\Omega_1 A_\phi(\rho)\Omega_n^\dagger+t^3\Omega_1\Omega_n^\dagger]}, \qquad (69)$$

where we have used $C_{x_{k+1}x_k}=2\pi\rho/n$, $\Omega(x_k)=\Omega_k$ and $\Omega_{n+1}=\Omega_1$.

For the subsequent evaluation it is convenient to introduce the matrix

$$Q(A_\phi)=\left(1+i\frac{2\pi}{n}\rho\,A_\phi(\rho)\right). \qquad (70)$$

In terms of $Q(A_\phi)$ the path integral (69) reads

$$W_{1/2}(\rho)=\lim_{n\to\infty}\int \ldots \int D\Omega_n D\Omega_{n-1}D\Omega_{n-2}\ldots D\Omega_1\, e^{\mathrm{tr}[t^3\Omega_n Q(A_\phi)\Omega^\dagger_{n-1}]} \times e^{\mathrm{tr}[t^3\Omega_{n-1}Q(A_\phi)\Omega^\dagger_{n-2}]}\ldots e^{\mathrm{tr}[t^3\Omega_2 Q(A_\phi)\Omega^\dagger_1]}\, e^{\mathrm{tr}[t^3\Omega_1 Q(A_\phi)\Omega^\dagger_n]}, \tag{71}$$

The integration over Ω_k we carry out with the help of Eq. (50) taken in the from

$$\int D\Omega_k\, e^{\mathrm{tr}[t^3\Omega_{k+1}Q(A_\phi)\Omega^\dagger_k+Q(A_\phi)\Omega^\dagger_{k-1}t^3\Omega_k]}=\sum_j \frac{a_j^2}{2j+1}\chi_j[(t^3)^2\Omega_{k+1}Q^2(A_\phi)\Omega^\dagger_{k-1}] \tag{72}$$

and the orthogonality relation (52). The coefficients a_j obey the completeness condition (59) with the constraint (66).

The number n may be both even or odd. Let n be an even number, then integrating over Ω_{n-1}, Ω_{n-3}, ..., Ω_1 by using Eq. (72) we obtain

$$W_{1/2}(\rho)=\lim_{n\to\infty}\int \ldots \int D\Omega_n D\Omega_{n-2}D\Omega_{n-4}\ldots D\Omega_2 \sum_{j_{n-1}}(2j_{n-1}+1)\left[\frac{a_{j_{n-1}}}{2j_{n-1}+1}\right]^2 \chi_{j_{n-1}}[(t^3)^2\Omega_n Q^2(A_\phi)\Omega^\dagger_{n-2}] \times\sum_{j_{n-3}}(2j_{n-3}+1)\left[\frac{a_{j_{n-3}}}{2j_{n-3}+1}\right]^2\chi_{j_{n-3}}[(t^3)^2\Omega_{n-2}Q^2(A_\phi)\Omega^\dagger_{n-4}]\ldots\sum_{j_4}(2j_4+1)\left[\frac{a_{j_4}}{2j_4+1}\right]^2 \times\chi_{j_4}[(t^3)^2\Omega_4 Q^2(A_\phi)\Omega^\dagger_2]\sum_{j_2}(2j_2+1)\left[\frac{a_{j_2}}{2j_2+1}\right]^2\chi_{j_2}[(t^3)^2\Omega_2 Q^2(A_\phi)\Omega^\dagger_n]. \tag{73}$$

The integration over Ω_n, Ω_{n-2}, ..., Ω_2 gives

$$\begin{aligned}W_J(\rho)&=\lim_{n\to\infty}\sum_j (2j+1)\left[\frac{a_j}{2j+1}\right]^n\int D\Omega_n\,\chi_j[(t^3)^n\Omega_n Q^n(A_\phi)\Omega^\dagger_n]\\&=\lim_{n\to\infty}\left(a_0^n+\sum_{j>0}\left[\frac{a_j}{2j+1}\right]^n\chi_j[(t^3)^n]\,\chi_j[Q^n(A_\phi)]\right)\\&=\lim_{n\to\infty}\sum_{j>0}\left[\frac{a_j}{2j+1}\right]^n\chi_j[(t^3)^n]\,\chi_j[Q^n(A_\phi)],\end{aligned} \tag{74}$$

where we have set $\lim_{n\to\infty}a_0^n=0$.

The integration over Ω_n we have carried out by means of Eq. (11). One can easily show that Eq. (74) is also valid for odd n, as well as Eq. (55). In this case due to the relation $\chi_j[(t^3)^n]=0$ we obtain again $W_{1/2}(\rho)=0$.

For even n we should use the relation $\chi_j[(t^3)^n]\simeq j^n$ at $n\to\infty$ which follows from Eq. (56). This reduces the RHS of Eq. (73) to the form

$$W_{1/2}(\rho)=\lim_{n\to\infty}\sum_{j>0}\left[\frac{ja_j}{2j+1}\right]^n\chi_j[Q^n(A_\phi)], \tag{75}$$

The evaluation of the character $\chi_j[Q^n(A_\phi)]$ for $n\to\infty$ runs as follows:

$$\begin{aligned}\chi_j[Q^n(A_\phi)]&=\chi_j\left[\left(1+i\frac{2\pi}{n}\rho A_\phi(\rho)\right)^n\right]\\&\simeq\chi_j[e^{i2\pi\rho A_\phi(\rho)}]\\&=\frac{\sin[(2j+1)\pi\mu(\rho)]}{\sin[\pi\mu(\rho)]}.\end{aligned} \tag{76}$$

Substituting Eq. (76) in Eq. (75) we obtain

$$W_{1/2}(\rho)=\lim_{n\to\infty}\sum_{j>0}\left[\frac{ja_j}{2j+1}\right]^n\frac{\sin[(2j+1)\pi\mu(\rho)]}{\sin[\pi\mu(\rho)]}. \tag{77}$$

Due to the constraint Eq. (66) the Wilson loop vanishes in the limit $n\to\infty$, $W_{1/2}(\rho)=0$. Thus, we have shown that the the path integral for Wilson loops suggested in Ref. [3] gives zero for a field configuration with a $Z(2)$ center vortex, $W_{1/2}(\rho)=0$, instead of the correct result $W_{1/2}(\rho)=\cos\pi\mu(\rho)$, Eq. (68).

We hope that the examples considered in Sec. IV and V demonstrate that the path integral representation for Wilson loops derived in Ref. [3] is erroneous. Nevertheless, in Sec. VI we evaluate explicitly the regularized evolution operator $Z_{\mathrm{Reg}}(R_2,R_1)$ suggested by Diakonov and Petrov for the representation of the Wilson loop in Ref. [3]. We show that this regularized evolution operator $Z_{\mathrm{Reg}}(R_2,R_1)$ has been evaluated incorrectly in Ref. [3]. The correct evaluation gives $Z_{\mathrm{Reg}}(R_2,R_1)=0$ which agrees with our results obtained above.

VI. PATH INTEGRAL FOR THE EVOLUTION OPERATOR $Z(R_2,R_1)$

As has been suggested in Ref. [3] the functional $Z(R_2,R_1)$ defined by (see Eq. (8) of Ref. [3])

$$Z(R_2,R_1)=\int_{R_1}^{R_2}DR(t)\exp\left(iT\int_{t_1}^{t_2}\mathrm{Tr}\,(iR\,\dot{R}\,\tau_3)\right),\quad (78)$$

where $\dot{R}=dR/dt$ and $T=1/2,1,3/2,\ldots$ is the color isospin quantum number, should be regularized by the analogy to an axial-symmetric top. The regularized expression has been defined in Eq. (9) of Ref. [3] by

$$Z_{\mathrm{Reg}}(R_2,R_1)=\int_{R_1}^{R_2}DR(t)\exp\left(i\int_{t_1}^{t_2}\left[\frac{1}{2}I_\perp\,(\Omega_1^2+\Omega_2^2)+\frac{1}{2}I_\parallel\,\Omega_3^2+T\,\Omega_3\right]\right),\quad (79)$$

where $\Omega_a=i\,\mathrm{Tr}(R\,\dot{R}\,\tau_a)$ are angular velocities of the top, τ_a are Pauli matrices $a=1,2,3$, $I_\perp$ and $I_\parallel$ are the moments of inertia of the top which should be taken to zero. According to the prescription of Ref. [3] one should take first the limit $I_\parallel\to 0$ and then $I_\perp\to 0$.

For the confirmation of the result, given in Eq. (13) of Ref. [3],

$$Z_{\mathrm{Reg}}(R_2,R_1)=(2T+1)\,D^T_{TT}(R_2R_1^\dagger),\quad (80)$$

where $D^T(U)$ is a Wigner rotational matrix in the representation T, the authors of Ref. [3] suggested to evaluate the evolution operator (79) explicitly by means of the discretization of the path integral over R. The discretized form of the evolution operator Eq. (79) is given by Eq. (14) of Ref. [3] and reads[1]

$$Z_{\mathrm{Reg}}(R_{N+1},R_0)=\lim_{\substack{N\to\infty\\ \delta\to 0}}\mathcal{N}\int\prod_{n=1}^{N}dR_n\exp\left[\sum_{n=0}^{N}\left(-i\frac{I_\perp}{2\delta}[(\mathrm{Tr}\,V_n\tau_1)^2+(\mathrm{Tr}\,V_n\tau_2)^2]-i\frac{I_\parallel}{2\delta}(\mathrm{Tr}\,V_n\tau_3)^2-T\,(\mathrm{Tr}\,V_n\tau_3)\right)\right],\quad (81)$$

where $R_n=R(s_n)$ with $s_n=t_1+n\,\delta$ and $\Omega_a=i\mathrm{Tr}\,(R_nR^\dagger_{n+1}\tau_a)/\delta$ is the discretized analogy of the angular velocities [3] and $V_n=R_nR^\dagger_{n+1}$ are the relative orientations of the top at neighbouring points. The normalization factor $\mathcal{N}$ is determined by

$$\mathcal{N}=\left(\frac{I_\perp}{2\pi i\,\delta}\sqrt{\frac{I_\parallel}{2\pi i\,\delta}}\right)^{N+1}.\quad (82)$$

[1]We are using the notations of Ref. [3].

[see Eq. (19) of Ref. [3]]. According to the prescription of Ref. [3] one should take the limits $\delta\to 0$ and $I_\parallel,I_\perp\to 0$ but keeping the ratios I_i/δ, where $(i=\parallel,\perp)$, much greater than unity, $I_i/\delta\gg 1$.

Let us rewrite the exponent of the integrand of Eq. (81) in equivalent form

$$Z_{\mathrm{Reg}}(R_{N+1},R_0)=\lim_{\substack{N\to\infty\\ \delta\to 0}}\left(\frac{I_\perp}{2\pi i\,\delta}\sqrt{\frac{I_\parallel}{2\pi i\,\delta}}\right)^{N+1}\int\prod_{n=1}^{N}dR_n\times\exp\left[\sum_{n=0}^{N}\left(-i\frac{I_\perp}{2\delta}(\mathrm{Tr}\,V_n\tau_a)^2-i\frac{I_\parallel-I_\perp}{2\delta}(\mathrm{Tr}\,V_n\tau_3)^2-T\,(\mathrm{Tr}\,V_n\tau_3)\right)\right].\quad (83)$$

Now let us show that if V_n is a rotation in the fundamental representation of $SU(2)$, so

$$(\mathrm{Tr}\,V_n\tau_a)^2=-4+(\mathrm{Tr}\,V_n)^2.\quad (84)$$

For this aim, first, recall that

$$\mathrm{Tr}\,(V_n\tau_a)=-\mathrm{Tr}\,(V_n^\dagger\tau_a).\quad (85)$$

Since V_n is a rotation matrix in the fundamental representation of $SU(2)$, it can be taken in the general standard parameterization given by Eq. (55). By virtue of the relation (85) we can rewrite $(\mathrm{Tr}\,V_n\tau_a)^2$ as follows:

$$\begin{aligned}(\mathrm{Tr}\,V_n\tau_a)^2&=-\mathrm{Tr}\,(V_n\tau_a)\mathrm{Tr}\,(V_n^\dagger\tau_a)\\&=-2\,\mathrm{Tr}\left(\left(V_n-\frac{1}{2}\mathrm{Tr}\,V_n\right)\left(V_n^\dagger-\frac{1}{2}\mathrm{Tr}\,V_n\right)\right)\\&=-2\,\mathrm{Tr}\,(R_nR^\dagger_{n+1}R_{n+1}R_n^\dagger)+(\mathrm{Tr}\,V_n)^2\\&=-2\,\mathrm{Tr}\,1+(\mathrm{Tr}\,V_n)^2=-4+(\mathrm{Tr}\,V_n)^2.\end{aligned}\quad (86)$$

By using the relation Eq. (84) we can recast the RHS of Eq. (83) into the form

$$Z_{\mathrm{Reg}}(R_{N+1},R_0)=\lim_{\substack{N\to\infty\\ \delta\to 0}}\left[\left(\frac{I_\perp}{2\pi i\,\delta}\sqrt{\frac{I_\parallel}{2\pi i\,\delta}}\right)^{N+1}\times\exp\left(iN(N+1)\frac{I_\perp}{\delta}\right)\right]\int\prod_{n=1}^{N}dR_n\times\exp\left[\sum_{n=0}^{N}\left(-i\frac{I_\perp}{2\delta}(\mathrm{Tr}\,V_n)^2-i\frac{I_\parallel-I_\perp}{2\delta}(\mathrm{Tr}\,V_n\tau_3)^2-T\,(\mathrm{Tr}\,V_n\tau_3)\right)\right].\quad (87)$$

Now let us proceed to the evaluation of the integrals over $R_n\,(n=1,2,\ldots,N)$. For this aim it is convenient to rewrite the RHS of Eq. (87) in the following form:

$$Z_{\rm Reg}(R_{N+1},R_0)=\lim_{\substack{N\to\infty\\ \delta\to 0}}\left[\left(\frac{I_\perp}{2\pi i\delta}\sqrt{\frac{I_\parallel}{2\pi i\delta}}\right)^{N+1}\exp\left(iN(N+1)\frac{I_\perp}{\delta}\right)\right]\int\int\ \dots\int\int dR_N\,dR_{N-1}\ \dots\ dR_2\,dR_1$$
$$\times\exp\left(-i\frac{I_\perp}{2\delta}[(\mathrm{Tr}\,R_NR^\dagger_{N+1})^2+(\mathrm{Tr}\,R_{N-1}R^\dagger_N)^2+\ \dots\ +(\mathrm{Tr}\,R_2R^\dagger_1)^2+(\mathrm{Tr}\,R_1R^\dagger_0)^2]\right.$$
$$-i\frac{I_\parallel-I_\perp}{2\delta}[(\mathrm{Tr}\,R_NR^\dagger_{N+1}\tau_3)^2+(\mathrm{Tr}\,R_{N-1}R^\dagger_N\tau_3)^2+\ \dots\ +(\mathrm{Tr}\,R_2R^\dagger_1\tau_3)^2+(\mathrm{Tr}\,R_1R^\dagger_0\tau_3)^2]$$
$$\left.-T[\mathrm{Tr}\,(R_NR^\dagger_{N+1}\tau_3)+\mathrm{Tr}\,(R_{N-1}R^\dagger_N\tau_3)+\ \dots\ +\mathrm{Tr}\,(R_2R^\dagger_1\tau_3)+\mathrm{Tr}\,(R_1R^\dagger_0\tau_3)]\right).\tag{88}$$

In the fundamental representation and the parametrization [18] (see Appendix) we have

$$\mathrm{Tr}\,V_n=\mathrm{Tr}\,(R_nR^\dagger_{n+1})=2\cos\frac{\beta_n}{2}\cos\frac{\beta_{n+1}}{2}\cos\left(\frac{\alpha_n+\gamma_n}{2}-\frac{\alpha_{n+1}+\gamma_{n+1}}{2}\right)+2\sin\frac{\beta_n}{2}\sin\frac{\beta_{n+1}}{2}\cos\left(\frac{\alpha_n-\gamma_n}{2}-\frac{\alpha_{n+1}-\gamma_{n+1}}{2}\right)$$
$$=2\cos\left(\frac{\beta_n-\beta_{n+1}}{2}\right)\cos\left(\frac{\alpha_n-\alpha_{n+1}}{2}\right)\cos\left(\frac{\gamma_n-\gamma_{n+1}}{2}\right)-2\cos\left(\frac{\beta_n+\beta_{n+1}}{2}\right)\sin\left(\frac{\alpha_n-\alpha_{n+1}}{2}\right)\sin\left(\frac{\gamma_n-\gamma_{n+1}}{2}\right),$$
$$\mathrm{Tr}\,(V_n\tau_3)=\mathrm{Tr}\,(R_nR^\dagger_{n+1}\tau_3)=-2i\cos\frac{\beta_n}{2}\cos\frac{\beta_{n+1}}{2}\sin\left(\frac{\alpha_n+\gamma_n}{2}-\frac{\alpha_{n+1}+\gamma_{n+1}}{2}\right)+2i\sin\frac{\beta_n}{2}\sin\frac{\beta_{n+1}}{2}$$
$$\times\sin\left(\frac{\alpha_n-\gamma_n}{2}-\frac{\alpha_{n+1}-\gamma_{n+1}}{2}\right)$$
$$=-2i\cos\left(\frac{\beta_n-\beta_{n+1}}{2}\right)\cos\left(\frac{\alpha_n-\alpha_{n+1}}{2}\right)\sin\left(\frac{\gamma_n-\gamma_{n+1}}{2}\right)-2i\cos\left(\frac{\beta_n+\beta_{n+1}}{2}\right)\sin\left(\frac{\alpha_n-\alpha_{n+1}}{2}\right)\cos\left(\frac{\gamma_n-\gamma_{n+1}}{2}\right).\tag{89}$$

The Haar measure R_n is defined by [see Eq. (A2)]:

$$DR_n=\frac{1}{8\pi^2}\sin\beta_n\,d\beta_n\,d\alpha_n\,d\gamma_n\,.\tag{90}$$

Due to the assumption $I_i/\delta\gg 1$, where $(i=\parallel,\perp)$, the integrals over R_n are concentrated around unit elements. Expanding $\mathrm{Tr}\,(V_n)$ and $\mathrm{Tr}\,(V_n\tau_3)$ around unit elements we get

$$\mathrm{Tr}\,V_n=\mathrm{Tr}\,(R_nR^\dagger_{n+1})=2-\frac{1}{4}(\beta_n-\beta_{n+1})^2-\frac{1}{4}(\alpha_n-\alpha_{n+1}+\gamma_n-\gamma_{n+1})^2,$$
$$\mathrm{Tr}\,(V_n\tau_3)=\mathrm{Tr}\,(R_nR^\dagger_{n+1}\tau_3)=-i(\alpha_n-\alpha_{n+1}+\gamma_n-\gamma_{n+1}).\tag{91}$$

For the subsequent integration it is convenient to make a change of variables

$$\frac{\alpha_n+\gamma_n}{2}\to\gamma_n\,,$$
$$\alpha_n-\gamma_n\to\alpha_n\,.\tag{92}$$

The Jacobian of this transformation is equal to unity. After this change of variables (91) reads

$$\mathrm{Tr}\,V_n=\mathrm{Tr}\,(R_nR^\dagger_{n+1})=2-\frac{1}{4}(\beta_n-\beta_{n+1})^2-(\gamma_n-\gamma_{n+1})^2,$$
$$\mathrm{Tr}\,(V_n\tau_3)=\mathrm{Tr}\,(R_nR^\dagger_{n+1}\tau_3)=-2i(\gamma_n-\gamma_{n+1}).\tag{93}$$

Since both $\mathrm{Tr}\,V_n$ and $\mathrm{Tr}\,(V_n\tau_3)$ do not depend on α_n, we can integrate out α_n. This changes only the Haar measure as follows:

$$DR_n=\frac{1}{4\pi}\beta_n\,d\beta_n\,d\gamma_n\,.\tag{94}$$

The integration over β_n and γ_n we will carry out in the limits $-\infty \leqslant \beta_n \leqslant \infty$ and $-\infty \leqslant \gamma_n \leqslant \infty$.

Substituting expansions (93) in the integrand of Eq. (88) we obtain

$$
\begin{aligned}
Z_{\rm Reg}(R_{N+1},R_0) &= \lim_{\substack{N\to\infty\\ \delta\to 0}} \left(\frac{I_\perp}{2\pi i\delta}\sqrt{\frac{I_\parallel}{2\pi i\delta}}\right)^{N+1}\left(\frac{1}{4\pi}\right)^N \int_{-\infty}^{\infty} d\gamma_N \int_{-\infty}^{\infty} d\beta_N\,\beta_N \int_{-\infty}^{\infty} d\gamma_{N-1}\int_{-\infty}^{\infty} d\beta_{N-1}\,\beta_{N-1}\,\ldots\int_{-\infty}^{\infty} d\gamma_2\\
&\quad\times \int_{-\infty}^{\infty} d\beta_2\,\beta_2 \int_{-\infty}^{\infty} d\gamma_1 \int_{-\infty}^{\infty} d\beta_1\,\beta_1 \exp\biggl(i\frac{I_\perp}{2\delta}[(\beta_{N+1}-\beta_N)^2+(\beta_N-\beta_{N-1})^2+\cdots+(\beta_2-\beta_1)^2\\
&\quad +(\beta_1-\beta_0)^2]+ i\frac{I_\parallel}{2\delta}[(\gamma_{N+1}-\gamma_N)^2+(\gamma_N-\gamma_{N-1})^2+\cdots+(\gamma_2-\gamma_1)^2+(\gamma_1-\gamma_0)^2]\\
&\quad -2\,i\,T[(\gamma_{N+1}-\gamma_N)+(\gamma_N-\gamma_{N-1})+\cdots+(\gamma_2-\gamma_1)+(\gamma_1-\gamma_0)]\biggr)\\
&= e^{-2\,i\,T\,(\gamma_{N+1}-\gamma_0)} \lim_{\substack{N\to\infty\\ \delta\to 0}} \left(\frac{I_\perp}{2\pi i\delta}\sqrt{\frac{I_\parallel}{2\pi i\delta}}\right)^{N+1}\left(\frac{1}{4\pi}\right)^N \int_{-\infty}^{\infty} d\gamma_N \int_{-\infty}^{\infty} d\beta_N\,\beta_N \int_{-\infty}^{\infty} d\gamma_{N-1}\\
&\quad\times \int_{-\infty}^{\infty} d\beta_{N-1}\,\beta_{N-1}\cdots\int_{-\infty}^{\infty} d\gamma_2 \int_{-\infty}^{\infty} d\beta_2\,\beta_2 \int_{-\infty}^{\infty} d\gamma_1 \int_{-\infty}^{\infty} d\beta_1\,\beta_1\\
&\quad\times \exp\biggl(i\frac{I_\perp}{2\delta}[(\beta_{N+1}-\beta_N)^2+(\beta_N-\beta_{N-1})^2+\cdots+(\beta_2-\beta_1)^2+(\beta_1-\beta_0)^2]\\
&\quad + i\frac{I_\parallel}{2\delta}[(\gamma_{N+1}-\gamma_N)^2+(\gamma_N-\gamma_{N-1})^2+\ \ldots+(\gamma_2-\gamma_1)^2+(\gamma_1-\gamma_0)^2]\biggr). \qquad (95)
\end{aligned}
$$

The integration over γ_n gives

$$
\begin{aligned}
&\int_{-\infty}^{\infty} d\gamma_N \int_{-\infty}^{\infty} d\gamma_{N-1}\cdots\int_{-\infty}^{\infty} d\gamma_2 \int_{-\infty}^{\infty} d\gamma_1 \exp\left(i\frac{I_\parallel}{2\delta}[(\gamma_{N+1}-\gamma_N)^2+(\gamma_N-\gamma_{N-1})^2+\cdots+(\gamma_2-\gamma_1)^2+(\gamma_1-\gamma_0)^2]\right)\\
&\quad=\sqrt{\frac{2\pi i\delta}{I_\parallel}\frac{1}{2}}\sqrt{\frac{2\pi i\delta}{I_\parallel}\frac{2}{3}}\cdots\sqrt{\frac{2\pi i\delta}{I_\parallel}\frac{N-1}{N}}\sqrt{\frac{2\pi i\delta}{I_\parallel}\frac{N}{N+1}}\exp\left(i\frac{I_\parallel}{2(N+1)\delta}(\gamma_{N+1}-\gamma_0)^2\right)\\
&\quad=\left(\sqrt{\frac{2\pi i\delta}{I_\parallel}}\right)^N\sqrt{\frac{1}{N+1}}\exp\left(i\frac{I_\parallel}{2(N+1)\delta}(\gamma_{N+1}-\gamma_0)^2\right). \qquad (96)
\end{aligned}
$$

By taking into account the normalization factor the result of the integration over γ_n reads

$$
\begin{aligned}
&\left(\sqrt{\frac{I_\parallel}{2\pi i\delta}}\right)^{N+1}\int_{-\infty}^{\infty} d\gamma_N \int_{-\infty}^{\infty} d\gamma_{N-1}\cdots\int_{-\infty}^{\infty} d\gamma_2 \int_{-\infty}^{\infty} d\gamma_1 \exp\biggl(i\frac{I_\parallel}{2\delta}[(\gamma_{N+1}-\gamma_N)^2+(\gamma_N-\gamma_{N-1})^2\\
&\quad+\cdots+(\gamma_2-\gamma_1)^2+(\gamma_1-\gamma_0)^2]\biggr)\\
&\quad=\sqrt{\frac{I_\parallel}{2\pi i(N+1)\delta}}\exp\left(i\frac{I_\parallel}{2(N+1)\delta}(\gamma_{N+1}-\gamma_0)^2\right)\\
&\quad=\sqrt{\frac{I_\parallel}{2\pi i\Delta t}}\exp\left(i\frac{I_\parallel}{2\Delta t}(\gamma_{N+1}-\gamma_0)^2\right), \qquad (97)
\end{aligned}
$$

where we have replaced $(N+1)\,\delta = t_2 - t_1 = \Delta t$. The obtained result is exact. By replacing $I_\parallel \to M$, $\gamma_{N+1}\to x_b$, $\gamma_0 \to x_a$ and $\Delta t \to (t_b - t_a)$ we arrive at the expression for the Green function, the evolution operator, of a free particle with a mass M given by Eq. (2.51) of Ref. [6].

Thus, after the integration over γ_n the evolution operator $Z_{\rm Reg}(R_{N+1},R_0)$ can be written in the form

$$
Z_{\rm Reg}(R_{N+1},R_0)=\sqrt{\frac{I_\parallel}{2\pi i\Delta t}}\exp\left(i\frac{I_\parallel}{2\Delta t}(\gamma_{N+1}-\gamma_0)^2\right) e^{-2\,i\,T\,(\gamma_{N+1}-\gamma_0)}\,F[I_\perp,\beta_{N+1},\beta_0], \qquad (98)
$$

where $F[I_\perp,\beta_{N+1},\beta_0]$ is a functional defined by the integrals over β_n

$$F[I_\perp,\beta_{N+1},\beta_0]=\lim_{\substack{N\to\infty\\ \delta\to 0}}\left(\frac{I_\perp}{2\pi i\delta}\right)^{N+1}\left(\frac{1}{4\pi}\right)^N\int_{-\infty}^{\infty}d\beta_N\,\beta_N\int_{-\infty}^{\infty}d\beta_{N-1}\,\beta_{N-1}\cdots\int_{-\infty}^{\infty}d\beta_2\,\beta_2\int_{-\infty}^{\infty}d\beta_1\,\beta_1$$
$$\times\exp\left(i\frac{I_\perp}{2\delta}[(\beta_{N+1}-\beta_N)^2+(\beta_N-\beta_{N-1})^2+\cdots+(\beta_2-\beta_1)^2+(\beta_1-\beta_0)^2]\right). \quad (99)$$

Formally we do not need to evaluate the functional $F[I_\perp,\beta_{N+1},\beta_0]$ explicitly. In fact, the functional $F[I_\perp,\beta_{N+1},\beta_0]$ should be a regular function of variables $I_\perp$, β_{N+1} and β_0 restricted in the limit $I_\perp\to 0$. Therefore, taking the limit $I_\parallel\to 0$ for the evolution operator $Z_{\rm Reg}(R_{N+1},R_0)$ defined by Eq. (98) we get

$$Z(R_2,R_1)=\lim_{I_\parallel,I_\perp\to 0}Z_{\rm Reg}(R_{N+1},R_0)=0. \quad (100)$$

This agrees with our results obtained in Secs. IV and V.

Nevertheless, in spite of this very definite result let us proceed to the explicit evaluation of the functional $F[I_\perp,\beta_{N+1},\beta_0]$. It is convenient to rewrite the integrand of Eq. (99) in the equivalent form

$$F[I_\perp,\beta_{N+1},\beta_0]=\lim_{\substack{N\to\infty\\ \delta\to 0}}\left(\frac{I_\perp}{2\pi i\delta}\right)\left(\frac{-1}{\pi}\right)^N\left(\frac{1}{4\pi}\right)^N\frac{\partial}{\partial j_1}\frac{\partial}{\partial j_2}\cdots\frac{\partial}{\partial j_{N-1}}\frac{\partial}{\partial j_N}\int_{-\infty}^{\infty}d\beta_N\int_{-\infty}^{\infty}d\beta_{N-1}\cdots\int_{-\infty}^{\infty}d\beta_2\int_{-\infty}^{\infty}d\beta_1$$
$$\times\exp\left(i\frac{I_\perp}{2\delta}[(\beta_{N+1}-\beta_N)^2+(\beta_N-\beta_{N-1})^2+\cdots+(\beta_2-\beta_1)^2+(\beta_1-\beta_0)^2\right.$$
$$\left.\left.+j_N\beta_N+j_{N-1}\beta_{N-1}+\cdots+j_2\beta_2+j_1\beta_1]\right)\right|_{j_N=j_{N-1}=\cdots=j_2=j_1=0}. \quad (101)$$

After k integrations we get

$$\int_{-\infty}^{\infty}d\beta_k\int_{-\infty}^{\infty}d\beta_{k-1}\cdots\int_{-\infty}^{\infty}d\beta_2\int_{-\infty}^{\infty}d\beta_1\exp\left(i\frac{I_\perp}{2\delta}[(\beta_{k+1}-\beta_k)^2+(\beta_k-\beta_{k-1})^2\right.$$
$$\left.+\cdots+(\beta_2-\beta_1)^2+(\beta_1-\beta_0)^2+j_k\beta_k+j_{k-1}\beta_{k-1}+\cdots+j_2\beta_2+j_1\beta_1]\right)$$
$$=\sqrt{\frac{2\pi i\delta}{I_\perp}\frac{1}{2}}\sqrt{\frac{2\pi i\delta}{I_\perp}\frac{2}{3}}\cdots\sqrt{\frac{2\pi i\delta}{I_\perp}\frac{k-1}{k}}\sqrt{\frac{2\pi i\delta}{I_\perp}\frac{k}{k+1}}\exp\left(i\frac{I_\perp}{2(k+1)\delta}(\beta_0-\beta_{k+1})^2\right)$$
$$\times\exp\left(i\frac{I_\perp}{\delta}\beta_{k+1}\left(\frac{k}{k+1}j_k+\frac{k}{k+1}\cdot\frac{k-1}{k}j_{k-1}+\frac{k}{k+1}\cdot\frac{k-1}{k}\cdot\frac{k-2}{k-1}j_{k-2}\right.\right.$$
$$\left.\left.+\cdots+\frac{k}{k+1}\cdot\frac{k-1}{k}\cdot\frac{k-2}{k-1}\cdots\frac{2}{3}j_2+\frac{k}{k+1}\cdot\frac{k-1}{k}\cdot\frac{k-2}{k-1}\cdots\frac{2}{3}\cdot\frac{1}{2}j_1\right)\right)$$
$$\times\exp\left(i\frac{I_\perp}{\delta}\left[-\frac{1}{2}\cdot\frac{1}{2}(j_1-\beta_0)^2+\frac{1}{2}\cdot\frac{1}{2}\beta_0^2-\frac{1}{2}\cdot\frac{2}{3}\left(j_2+\frac{1}{2}j_1-\frac{1}{2}\beta_0\right)^2+\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{2^2}\beta_0^2\right.\right.$$
$$-\frac{1}{2}\cdot\frac{3}{4}\left(j_3+\frac{2}{3}j_2+\frac{2}{3}\cdot\frac{1}{2}j_1-\frac{1}{3}\beta_0\right)^2+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{3^2}\beta_0^2-\frac{1}{2}\cdot\frac{4}{5}\left(j_4+\frac{3}{4}j_3+\frac{3}{4}\cdot\frac{2}{3}j_2\right.$$
$$\left.+\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}j_1-\frac{1}{4}\beta_0\right)^2+\frac{1}{2}\cdot\frac{4}{5}\cdot\frac{1}{4^2}\beta_0^2-\frac{1}{2}\cdot\frac{5}{6}\left(j_5+\frac{4}{5}j_4+\frac{4}{5}\cdot\frac{3}{4}j_3+\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}j_2\right.$$
$$\left.+\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}j_1-\frac{1}{5}\beta_0\right)^2+\frac{1}{2}\cdot\frac{5}{6}\cdot\frac{1}{5^2}\beta_0^2-\ \ldots-\frac{1}{2}\cdot\frac{k}{k+1}\left(j_k+\frac{k-1}{k}j_{k-1}\right.$$
$$\left.+\frac{k-1}{k}\cdot\frac{k-2}{k-1}j_{k-2}+\ \ldots+\frac{k-1}{k}\cdot\frac{k-2}{k-1}\cdots\frac{2}{3}j_2+\frac{k-1}{k}\cdot\frac{k-2}{k-1}\cdots\frac{2}{3}\cdot\frac{1}{2}j_1-\frac{1}{k}\beta_0\right)^2$$
$$\left.\left.+\frac{1}{2}\cdot\frac{k}{k+1}\cdot\frac{1}{k}^2\beta_0^2\right]\right). \quad (102)$$

By performing N integrations we obtain

$$\int_{-\infty}^{\infty} d\beta_N \int_{-\infty}^{\infty} d\beta_{N-1} \cdots \int_{-\infty}^{\infty} d\beta_2 \int_{-\infty}^{\infty} d\beta_1 \exp\biggl(i\frac{I_\perp}{2\delta}[(\beta_{N+1}-\beta_N)^2+(\beta_N-\beta_{N-1})^2$$
$$+\cdots+(\beta_2-\beta_1)^2+(\beta_1-\beta_0)^2+j_N\beta_N+j_{N-1}\beta_{N-1}+\cdots+j_2\beta_2+j_1\beta_1]\biggr)$$
$$=\left(\sqrt{\frac{2\pi i\delta}{I_\perp}}\right)^{N+1}\sqrt{\frac{I_\perp}{2\pi i\Delta t}}\exp\left(i\frac{I_\perp}{2\Delta t}(\beta_{N+1}-\beta_0)^2\right)\exp\biggl(i\frac{I_\perp}{\delta}\beta_{N+1}\biggl(\frac{N}{N+1}j_N+\frac{N}{N+1}\cdot\frac{N-1}{N}j_{N-1}$$
$$+\frac{N}{N+1}\cdot\frac{N-1}{N}\cdot\frac{N-2}{N-1}j_{N-2}+\cdots+\frac{N}{N+1}\cdot\frac{N-1}{N}\cdot\frac{N-2}{N-1}\cdots\frac{2}{3}j_2$$
$$+\frac{N}{N+1}\cdot\frac{N-1}{N}\cdot\frac{N-2}{N-1}\cdots\frac{2}{3}\cdot\frac{1}{2}j_1\biggr)\biggr)\exp\biggl(i\frac{I_\perp}{\delta}\biggl[-\frac{1}{2}\cdot\frac{1}{2}(j_1-\beta_0)^2+\frac{1}{2}\cdot\frac{1}{2}\beta_0^2$$
$$-\frac{1}{2}\cdot\frac{2}{3}\left(j_2+\frac{1}{2}j_1-\frac{1}{2}\beta_0\right)^2+\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{2^2}\beta_0^2-\frac{1}{2}\cdot\frac{3}{4}\left(j_3+\frac{2}{3}j_2+\frac{2}{3}\cdot\frac{1}{2}j_1-\frac{1}{3}\beta_0\right)^2+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{3^2}\beta_0^2$$
$$-\frac{1}{2}\cdot\frac{4}{5}\left(j_4+\frac{3}{4}j_3+\frac{3}{4}\cdot\frac{2}{3}j_2+\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}j_1-\frac{1}{4}\beta_0\right)^2+\frac{1}{2}\cdot\frac{4}{5}\frac{1}{4^2}\beta_0^2-\cdots-\frac{1}{2}\frac{N}{N+1}\biggl(j_N+\frac{N-1}{N}j_{N-1}$$
$$+\frac{N-1}{N}\cdot\frac{N-2}{N-1}j_{N-2}+\cdots+\frac{N-1}{N}\cdot\frac{N-2}{N-1}\cdots\frac{2}{3}j_2+\frac{N-1}{N}\cdot\frac{N-2}{N-1}\cdots\frac{2}{3}\cdot\frac{1}{2}j_1-\frac{1}{N}\beta_0\biggr)^2+\frac{1}{2}\cdot\frac{N}{N+1}\cdot\frac{1}{N^2}\beta_0^2\biggr]\biggr), \tag{103}$$

where we have replaced $(N+1)\,\delta=t_2-t_1=\Delta t$.

Now we can evaluate the derivatives with respect to $j_1, j_2, \ldots, j_{N-1}, j_N$. Due to the constraint $I_\perp/\delta \gg 1$ we can keep only the leading order contributions in powers of $I_\perp/\delta \gg 1$. The result reads

$$\frac{\partial}{\partial j_1}\frac{\partial}{\partial j_2}\cdots\frac{\partial}{\partial j_{N-1}}\frac{\partial}{\partial j_N}\int_{-\infty}^{\infty} d\beta_N \int_{-\infty}^{\infty} d\beta_{N-1} \cdots \int_{-\infty}^{\infty} d\beta_2 \int_{-\infty}^{\infty} d\beta_1 \exp\biggl(i\frac{I_\perp}{2\delta}[(\beta_{N+1}-\beta_N)^2+(\beta_N-\beta_{N-1})^2$$
$$+\cdots+(\beta_2-\beta_1)^2+(\beta_1-\beta_0)^2+j_N\beta_N+j_{N-1}\beta_{N-1}+\cdots+j_2\beta_2+j_1\beta_1]\biggr)\biggr|_{j_N=j_{N-1}=\ldots=j_2=j_1=0}$$
$$=\left(\sqrt{\frac{2\pi i\delta}{I_\perp}}\right)^{N+1}\sqrt{\frac{I_\perp}{2\pi i\Delta t}}\exp\left(i\frac{I_\perp}{2\Delta t}(\beta_{N+1}-\beta_0)^2\right)\left(\frac{iI_\perp}{\delta}\right)^N\left(\beta_{N+1}\frac{1}{N+1}+\beta_0\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\frac{1}{N(N+1)}\right]\right)$$
$$\times\left(\beta_{N+1}\frac{2}{N+1}+\beta_0\left[\frac{2}{2\cdot 3}+\frac{2}{3\cdot 4}+\ldots+\frac{2}{N(N+1)}\right]\right)\left(\beta_{N+1}\frac{3}{N+1}+\beta_0\left[\frac{3}{3\cdot 4}+\cdots+\frac{3}{N(N+1)}\right]\right)$$
$$\times\left(\beta_{N+1}\frac{4}{N+1}+\beta_0\left[\frac{4}{4\cdot 5}+\cdots+\frac{4}{N(N+1)}\right]\right)\ldots\left(\beta_{N+1}\frac{N}{N+1}+\beta_0\frac{N}{N(N+1)}\right)$$
$$=\left(\sqrt{\frac{2\pi i\delta}{I_\perp}}\right)^{N+1}\sqrt{\frac{I_\perp}{2\pi i\Delta t}}\exp\left(i\frac{I_\perp}{2\Delta t}(\beta_{N+1}-\beta_0)^2\right)\left(\frac{iI_\perp}{\delta}\right)^N\left(\beta_{N+1}\frac{1}{N+1}+\beta_0\frac{N}{N+1}\right)\left(\beta_{N+1}\frac{2}{N+1}+\beta_0\frac{N-1}{N+1}\right)$$
$$\times\left(\beta_{N+1}\frac{3}{N+1}+\beta_0\frac{N-2}{N+1}\right)\times\cdots\times\left(\beta_{N+1}\frac{N}{N+1}+\beta_0\frac{1}{N+1}\right)$$
$$=\left(\sqrt{\frac{2\pi i\delta}{I_\perp}}\right)^{N+1}\sqrt{\frac{I_\perp}{2\pi i\Delta t}}\exp\left(i\frac{I_\perp}{2\Delta t}(\beta_{N+1}-\beta_0)^2\right)\left(\frac{iI_\perp}{\delta}\right)^N\prod_{k=1}^{N}\left(\beta_{N+1}\frac{k}{N+1}+\beta_0\frac{N+1-k}{N+1}\right). \tag{104}$$

Substituting Eq. (104) in Eq. (101) we obtain the functional $F[I_\perp, \beta_{N+1}, \beta_0]$

$$F[I_\perp, \beta_{N+1}, \beta_0] = \sqrt{\frac{I_\perp}{2\pi i \Delta t}} \exp\left(i \frac{I_\perp}{2\Delta t}(\beta_{N+1} - \beta_0)^2\right) \times \lim_{\substack{N\to\infty \\ \delta\to 0}} \left(\frac{1}{2\pi}\right)^N \left(\sqrt{\frac{I_\perp}{2\pi i \delta}}\right)^{N+1} \prod_{k=1}^{N} \left(\beta_{N+1}\frac{k}{N+1} + \beta_0 \frac{N+1-k}{N+1}\right). \tag{105}$$

In order to understand the behavior of the functional $F[I_\perp, \beta_{N+1}, \beta_0]$ in the limit $N\to\infty$ we suggest evaluating the product

$$\Pi[\beta_{N+1}, \beta_0] = \prod_{k=1}^{N} \left(\beta_{N+1}\frac{k}{N+1} + \beta_0 \frac{N+1-k}{N+1}\right) \tag{106}$$

at $N \gg 1$ by using the ζ regularization. In the ζ regularization the evaluation of $\Pi[\beta_{N+1}, \beta_0]$ runs the following way

$$\begin{aligned}
\ln \Pi[\beta_{N+1}, \beta_0] &= \sum_{k=1}^{N} \ln\left(\beta_{N+1}\frac{k}{N+1} + \beta_0\frac{N+1-k}{N+1}\right) = \sum_{k=1}^{N} (-1)\frac{d}{ds}\left(\beta_{N+1}\frac{k}{N+1} + \beta_0\frac{N+1-k}{N+1}\right)^{-s}\Bigg|_{s=0} \\
&= -\frac{d}{ds}\sum_{k=1}^{N}\int_0^\infty \frac{dz}{\Gamma(s)} \exp\left[-\left(\beta_{N+1}\frac{k}{N+1} + \beta_0\frac{N+1-k}{N+1}\right)z\right] z^{s-1}\Bigg|_{s=0} \\
&= -\frac{d}{ds}\int_0^\infty \frac{dz}{\Gamma(s)}\left[\frac{e^{-\beta_0 z} - e^{-\beta_{N+1} z}}{1 - \exp\left(-\dfrac{\beta_{N+1}-\beta_0}{N+1}z\right)} - e^{-\beta_0 z}\right] z^{s-1}\Bigg|_{s=0} \\
&= -(N+1)\frac{d}{ds}\int_0^\infty \frac{dz}{\Gamma(s)} \frac{e^{-\beta_0 z} - e^{-\beta_{N+1} z}}{\beta_{N+1} - \beta_0} z^{s-2}\Bigg|_{s=0} \\
&= -(N+1)\frac{d}{ds}\left[\frac{1}{s-1}\frac{\beta_0^{1-s} - \beta_{N+1}^{1-s}}{\beta_{N+1} - \beta_0}\right]_{s=0} \\
&= -(N+1)\left[1 - \frac{\beta_{N+1}\ln\beta_{N+1} - \beta_0 \ln\beta_0}{\beta_{N+1} - \beta_0}\right] \\
&= -(N+1)\frac{\beta_{N+1}\ln\dfrac{e}{\beta_{N+1}} - \beta_0\ln\dfrac{e}{\beta_0}}{\beta_{N+1} - \beta_0}.
\end{aligned} \tag{107}$$

Thus, the function $\Pi[\beta_{N+1}, \beta_0]$ is defined by

$$\Pi[\beta_{N+1}, \beta_0] = \exp\left(-(N+1)\frac{\beta_{N+1}\ln\dfrac{e}{\beta_{N+1}} - \beta_0\ln\dfrac{e}{\beta_0}}{\beta_{N+1} - \beta_0}\right), \tag{108}$$

where $e = 2.71828\ldots$. Due to the constraint $I_\perp/\delta \gg 1$ the Euler angles β_{N+1} and β_0 are less than unity and the ratio in Eq. (108) is always positive

$$\frac{\beta_{N+1}\ln\dfrac{e}{\beta_{N+1}} - \beta_0\ln\dfrac{e}{\beta_0}}{\beta_{N+1} - \beta_0} > 0. \tag{109}$$

The functional $F[I_\perp, \beta_{N+1}, \beta_0]$ is then defined by

$$\begin{aligned}
F[I_\perp, \beta_{N+1}, \beta_0] = &\sqrt{\frac{I_\perp}{2\pi i \Delta t}} \exp\left(i\frac{I_\perp}{2\Delta t}(\beta_{N+1} - \beta_0)^2\right) \lim_{\substack{N\to\infty \\ \delta\to 0}} \left(\frac{1}{2\pi}\right)^N \left(\sqrt{\frac{I_\perp}{2\pi i\delta}}\right)^{N+1} \\
&\times \exp\left(-(N+1)\frac{\beta_{N+1}\ln\dfrac{e}{\beta_{N+1}} - \beta_0\ln\dfrac{e}{\beta_0}}{\beta_{N+1} - \beta_0}\right).
\end{aligned} \tag{110}$$

Thus $F[I_\perp,\beta_{N+1},\beta_0]$ vanishes in the limit $N\to\infty$. This result retains itself even if we change the normalization factor of the evolution operator

$$\mathcal{N}=\left(\frac{I_\perp}{2\pi i\delta}\sqrt{\frac{I_\parallel}{2\pi i\delta}}\right)^{N+1}\to(2\pi)^N\left(\sqrt{\frac{I_\parallel}{I_\perp}}\right)^{N+1}. \tag{111}$$

The renormalized functional $F[I_\perp,\beta_{N+1},\beta_0]$, defined by

$$F[I_\perp,\beta_{N+1},\beta_0]=\sqrt{\frac{I_\perp}{2\pi i\Delta t}}\exp\left(i\frac{I_\perp}{2\Delta t}(\beta_{N+1}-\beta_0)^2\right)\lim_{N\to\infty}\exp\left(-(N+1)\frac{\beta_{N+1}\ln\dfrac{e}{\beta_{N+1}}-\beta_0\ln\dfrac{e}{\beta_0}}{\beta_{N+1}-\beta_0}\right), \tag{112}$$

vanishes in the limit $N\to\infty$ since the Euler angles β_{N+1} and β_0 are small compared with unity due to the constraint $I_\perp/\delta\gg 1$ [3]. The vanishing of the functional $F[I_\perp,\beta_{N+1},\beta_0]$ in the limit $N\to\infty$ agrees with our results obtained in Secs. IV and V.

Substituting Eq. (112) in Eq. (98) we obtain

$$Z_{\rm Reg}(R_{N+1},R_0)=0. \tag{113}$$

This leads to the vanishing of the evolution operator $Z(R_2,R_1)$ given by Eq. (78) or Eq. (8) of Ref. [3], $Z(R_2,R_1)=0$.

Thus, the evolution operator $Z(R_2,R_1)$ suggested in Ref. [3] for the description of Wilson loops in terms of path integrals over gauge degrees of freedom is equal to zero identically. This agrees with our results obtained in Secs. IV and V. As we have shown the vanishing of $Z(R_2,R_1)$ does not depend on the specific regularization and discretization of the path integral. In fact, this is an intrinsic property of the path integral given by Eq. (78) that becomes obvious if the evaluation is carried out correctly.

VII. EVOLUTION OPERATOR $Z(R_2,R_1)$ AND SHIFT OF ENERGY LEVELS OF AN AXIAL-SYMMETRIC TOP

In this section we criticize the analysis of the evolution operator $Z(R_2,R_1)$ carried out by Diakonov and Petrov via *the canonical quantization of the axial–symmetric top* [see Eq. (12) of Ref. [3]]. Below we use the notation of Ref. [3].

The parallel transport operator

$$W_{\alpha\beta}(t_2,t_1)=\left[P\exp\left(i\int_{x(t_1)}^{x(t_2)}A^a_\mu(x)\,T^a\,dx_\mu\right)\right]_{\alpha\beta}=\left[P\exp\left(i\int_{t_1}^{t_2}A(t)\,dt\right)\right]_{\alpha\beta}, \tag{114}$$

where $A(t)=A^a_\mu(x)\,T^a\,dx_\mu/dt$ is a tangent component of the Yang-Mills field and $T^a\,(a=1,2,3)$ are the generators of $SU(2)$ group in the representation T, has been reduced to the form

$$W_{\alpha\beta}(t_2,t_1)=D^T_{\alpha\beta}(U(t_2)U^\dagger(t_1)) \tag{115}$$

[see Eq. (5) of Ref. [3]] due to the statement [3]: *The potential $A(t)$ along a given curve can be always written as a "pure gauge"*

$$A_{\alpha\beta}(t)=i\,D^T_{\alpha\gamma}(U(t))\frac{d}{dt}D^T_{\gamma\beta}(U^\dagger(t)) \tag{116}$$

[see Eq. (4) of Ref. [3]].

By using the parallel transport operator Eq. (115) the Wilson loop $W_T(C)$ in the representation T has been defined by

$$W_T(C)=\sum_\alpha W_{\alpha\alpha}(t_2,t_1)=\sum_\alpha D^T_{\alpha\alpha}(U(t_2)U^\dagger(t_1)) \tag{117}$$

[see Eq. (25) of Ref. [3]]. In terms of the evolution operator $Z(R_2,R_1)$ given by Eq. (78) [see Eq. (8) of Ref. [3]] the parallel transport operator $W_{\alpha\beta}(t_2,t_1)$ has been recast into the form

$$W^{\rm DP}_{\alpha\beta}(t_2,t_1)=\int\int dR_1\,dR_2\sum_{T',m}(2T'+1)\,D^{T'}_{\alpha m}(U(t_2)R_2^\dagger)\times D^{T'}_{m\beta}(R_1U^\dagger(t_1))\,Z(R_2,R_1), \tag{118}$$

where the index DP means that the parallel transport operator is taken in the Diakonov-Petrov (DP) representation. The Wilson loop $W^{\rm DP}_T(C)$ in the DP representation reads

$$W^{\rm DP}_T(C)=\int\int dR_1\,dR_2\sum_{T',m,\alpha}(2T'+1)\,D^{T'}_{\alpha m}(U(t_2)R_2^\dagger)\times D^{T'}_{m\alpha}(R_1U^\dagger(t_1))\,Z(R_2,R_1). \tag{119}$$

Of course, if the DP representation were correct we should get $W^{\rm DP}_T(C)=W_T(C)$, where $W_T(C)$ is determined by Eq. (117).

The regularized evolution operator $Z_{\rm Reg}(R_2,R_1)$ given by Eq. (80) [see also Eq. (9) of Ref. [3]] can be represented in the form of *a sum over possible intermediate states*, i.e. eigenfunctions of the axial-symmetric top

$$Z_{\rm Reg}(R_2,R_1)=\sum_{J,m,k}(2J+1)\,D^J_{mk}(R_2)D^J_{km}(R_1^\dagger)\times e^{-i(t_2-t_1)\,E_{Jm}}, \qquad (120)$$

[see Eq. (12) of Ref. [3]], where E_{Jm} are the eigenvalues of the Hamiltonian of the axial-symmetric top

$$E_{Jm}=\frac{J(J+1)-m^2}{2I_\perp}+\frac{(m-T)^2}{2I_\parallel} \qquad (121)$$

[see Eq. (11) of Ref. [3]].

As has been stated in Ref. [3]: *If we now take to zero $I_{\perp,\parallel}\to 0$ (first $I_\parallel$, then $I_\perp$) we see that in the sum (12) only the lowest energy intermediate state survives with $m=J=T$. The resulting phase factor from the lowest energy state can be absorbed in the normalization factor in Eq. (9) since that corresponds to a shift in the energy scale.*

The statement concerning the possibility to absorb the fluctuating factor $\exp[-i(t_2-t_1)\,T/2I_\perp]$ in the normalization of the path integral representing the evolution operator is the main one allowing the RHS of Eqs. (118) and (119) to escape from the vanishing in the limit $I_\perp\to 0$.

In reality such a removal of the fluctuating factor is prohibited since this leads to the change of the starting symmetry of the system from $SU(2)$ to $U(2)$. In order to make this more transparent we suggest to insert $Z_{\rm Reg}(R_2,R_1)$ of Eq. (120) into Eq. (119) and to express the Wilson loop $W_T^{\rm DP}(C)$ in terms of *a sum over possible intermediate states*, the eigenfunctions of the axial-symmetric top. The main idea of this substitution is the following: as the Wilson loop is a physical quantity which can be measured, all irrelevant normalization factors should be canceled for the evaluation of it. Therefore, if the oscillating factor $\exp[-i(t_2-t_1)\,T/2I_\perp]$ can be really removed by a renormalization of something, the Wilson loop should not depend on this factor.

Substituting Eq. (120) in Eq. (119) and integrating over R_1 and R_2 we obtain the following expansion for the parallel transport operator in the DP representation:

$$W^{\rm DP}_{\alpha\beta}(t_2,t_1)=\sum_{T'}\sum_{m=-T'}^{T'}D^{T'}_{\alpha\beta}(U(t_2)U^\dagger(t_1))\,e^{-i(t_2-t_1)\,E_{T'm}}. \qquad (122)$$

Setting $\alpha=\beta$ and summing over α we get the DP representation for Wilson loops

$$W_T^{\rm DP}(C)=\sum_\alpha W^{\rm DP}_{\alpha\alpha}(t_2,t_1)=\sum_{T'}\sum_{m=-T'}^{T'}D^{T'}_{\alpha\alpha}(U(t_2)U^\dagger(t_1))\,e^{-i(t_2-t_1)\,E_{T'm}}. \qquad (123)$$

Due to the definition (117) the RHS of Eq. (123) can be rewritten in the form

$$W_T^{\rm DP}(C)=\sum_\alpha W^{\rm DP}_{\alpha\alpha}(t_2,t_1)=\sum_{T'}\sum_{m=-T'}^{T'}W_{T'}(C)\,e^{-i(t_2-t_1)\,E_{T'm}}, \qquad (124)$$

where $W_{T'}(C)$ is the Wilson loop in the T' representation determined by Eq. (117). The relation (124) agrees to some extent with our expansion given by Eq. (114).

Following Ref. [3] and taking the limit $I_\parallel\to 0$ we obtain $m=T$. This reduces the RHS of Eq. (124) to the form

$$W_T^{\rm DP}(C)=\sum_{T'}W_{T'}(C)\exp\left[-i(t_2-t_1)\frac{T'(T'+1)-T^2}{2I_\perp}\right]. \qquad (125)$$

Now according to the prescription of Ref. [3] we should take the limit $I_\perp\to 0$. Following again Ref. [3] and setting $T'=T$ we arrive at the relation

$$W_T^{\rm DP}(C)=W_T(C)\exp\left[-i(t_2-t_1)\frac{T}{2I_\perp}\right]=\sum_\alpha D^T_{\alpha\alpha}(U(t_2)U^\dagger(t_1))\exp\left[-i(t_2-t_1)\frac{T}{2I_\perp}\right]. \qquad (126)$$

Thus, Wilson loops in the DP representation differ from original Wilson loops by the oscillating factor $\exp[i(t_2-t_1)\,T/2I_\perp]$. The only possibility to remove the oscillating factor $\exp[-i(t_2-t_1)\,T/2I_\perp]$ is to absorb this phase factors in the matrices $U(t_2)$ and $U^\dagger(t_1)$ which describe the degrees of freedom of the gauge potential $A(t)$ via the relation (115). This yields the changes

$$U(t_2)\to\bar U(t_2)=U(t_2)\,e^{i\,t_2\,T/2I_\perp},\qquad U^\dagger(t_1)\to\bar U^\dagger(t_1)=U^\dagger(t_1)\,e^{-\,i\,t_1\,T/2I_\perp}. \qquad (127)$$

However, the matrices $\bar U(t_2)$ and $\bar U^\dagger(t_1)$ are now the elements of the $U(2)$ group but not of $SU(2)$. Thus, the shift of the energy level of the ground state of the axial-symmetric top suggested by Diakonov and Petrov in order to remove the oscillating factor changes crucially the starting symmetry of the theory from $SU(2)$ to $U(2)$. Since the former is not allowed the oscillating factor $\exp[-i(t_2-t_1)\,T/2I_\perp]$ cannot be removed. As a result in the limit $I_\perp\to 0$ we obtain

$$W_T^{\rm DP}(C)=0. \qquad (128)$$

The vanishing of Wilson loops in the DP representation agrees with our results obtained in Secs. IV, V, and VI and confirms our claim that this path integral representation of Wilson loops is incorrect.

VIII. THE NON-ABELIAN STOKES THEOREM

The derivation of the area-law falloff promoted great interests in the non-Abelian Stokes theorem expressing the exponent of Wilson loops in terms of a surface integral over the 2-dimensional surface S with the boundary $C=\partial S$ [18]

$$\mathrm{tr}\,\mathcal{P}_C e^{i\,g\,\oint_C dx_\mu\,A_\mu(x)} = \mathrm{tr}\,\mathcal{P}_S\, e^{i\,g\,1/2\int\int_S d\sigma_{\mu\nu}(y)\,U(C_{xy})\,G_{\mu\nu}(y)\,U(C_{yx})}, \qquad (129)$$

where $\mathcal{P}_S$ is the surface ordering operator [18], $d\sigma^{\mu\nu}(y)$ is a 2-dimensional surface element in 4-dimensional space-time, x is a current point on the contour C, i.e. $x\in C$, y is a point on the surface S, i.e. $y\in S$, and $G_{\mu\nu}(y)=\partial_\mu A_\nu(y)-\partial_\nu A_\mu(y)-ig[A_\mu(y),A_\nu(y)]$ is the field strength tensor. The procedure for the derivation of the non-Abelian Stokes theorem in the form of Eq. (129) contains a summation of contributions of closed paths around infinitesimal areas and these paths are linked to the reference point x on the contour C via parallel transport operators. The existence of closed paths linked to the references point x on the contour C is a *necessary* and a *sufficient* condition for the derivation of the non-Abelian Stokes theorem Eq. (129).

Due to the absence of closed paths it is rather clear that the path integral representation for Wilson loops cannot be applied to the derivation of the non-Abelian Stokes theorem. In fact, the evaluation of the path integral over gauge degrees of freedom demands the decomposition of the closed contour C into a set of infinitesimal segments which can be never closed. Let us prove this statement by assuming the converse. Suppose that by representing the path integral over gauge degrees of freedom in the form of the n-dimensional integral (15) we have a closed segment. Let the segment $C_{x_kx_{k-1}}$ be closed and the point x' belong to the segment $C_{x_kx_{k-1}}$, $x'\in C_{x_kx_{k-1}}$. By using Eq. (10) we can represent the character $\chi[U^\Omega_r(C_{x_kx_{k-1}})]$ by

$$\begin{aligned}\chi[U^\Omega_r(C_{x_kx_{k-1}})]&=\chi[U^\Omega_r(C_{x_ix'})U^\Omega_r(C_{x'x_{i-1}})]=\chi[\Omega(x_i)U_r(C_{x_ix'})U_r(C_{x'x_{i-1}})\Omega(x_{i-1})]\\&=d_r\int D\Omega_r(x')\chi[\Omega_r(x_i)U_r(C_{x_ix'})\Omega^\dagger_r(x')]\,\chi[\Omega_r(x')U_r(C_{x'x_{i-1}})\Omega(x_{i-1})]\\&=d_r\int D\Omega_r(x')\chi[U^\Omega_r(C_{x_ix'})]\,\chi[U^\Omega_r(C_{x'x_{i-1}})].\end{aligned}\qquad (130)$$

This transforms a $(n-1)$-dimensional integral with one closed infinitesimal segment into a n-dimensional integral without closed segments. Since finally n tends to infinity there is no closed segments for the representation for the path integral in the form of a $(n-1)$-dimensional integral. As this statement is general and valid for any path integral representation of Wilson loops, so one can conclude that no further non-Abelian Stokes theorem can be derived within any path integral approach to Wilson loops.

IX. CONCLUSION

By using well defined properties of group characters we have shown that the path integral over gauge degrees of freedom of the Wilson loop which was used in Eq. (2.13) of Ref. [11] for a lattice evaluation of the average value of Wilson loops can be derived in continuum space-time in non-Abelian gauge theories with the gauge group $SU(N)$. The resultant integrand of the path integral contains a phase factor which is not projected onto Abelian degrees of freedom of non-Abelian gauge fields and differs substantially from the representation given in Ref. [3]. The important point of our representation is the summation over all states of the given irreducible representation r of $SU(N)$. For example, in $SU(2)$ the phase factor is summed over all values of the colourmagnetic quantum number m_j of the irreducible representation j of color charges. This contradicts Eq. (23) of Ref. [3], where only term with the highest value of the color magnetic quantum number $m_j=j$ are taken into account and the other $2j$ terms are lost. This loss is caused by an artificial regularization procedure applied in Ref. [3] for the definition of the path integral over gauge degrees of freedom.

As has been stated by Diakonov and Petrov in Ref. [3] the path integral over gauge degrees of freedom representing Wilson loops *is not of the Feynman type, therefore, it depends explicitly on how one ''understands'' it, i.e. how it is discretized and regularized*. In order to *understand* the path integral over gauge degrees of freedom Diakonov and Petrov [3] suggested a regularization procedure drawing an analogy between gauge degrees of freedom and dynamical variables of the axial-symmetric top with moments of inertia $I_\perp$ and $I_\parallel$. The final expression for the path integral of the Wilson loop has been obtained in the limit $I_\perp, I_\parallel\to 0$.

In order to make the incorrectness of this expression more transparent we have evaluated the path integral for specific gauge field configurations (i) a pure gauge field and (ii) $Z(2)$ center vortices with spatial azimuthal symmetry. The direct evaluation of path integrals representing Wilson loops for these gauge field configurations has given the value zero for both cases. These results do not agree with the correct values.

One can show that Eq. (77) can be generalized for any contour of a Wilson loop in $SU(2)$:

$$W_{1/2}(C)=\int\prod_{x\in C}D\Omega(x)\,e^{ig\oint_C dx_\mu\,\mathrm{tr}[t^3A^\Omega_\mu(x)]}$$
$$=\lim_{n\to\infty}\sum_{j>0}\left[\frac{ja_j}{2j+1}\right]^n(2j+1)W_j(C)=0, \tag{131}$$

where $W_j(C)$ in the RHS is defined by Eq. (9) in terms of the path-ordering operator $\mathcal{P}_C$. Further, the result (131) can be extended to any irreducible representation of $SU(2)$. Thus, we argue that the path integral suggested in Ref. [3] to represent Wilson loops is identically zero for Wilson loops independent on the gauge field configuration, the shape of the contour C and the irreducible representation of $SU(2)$.

This statement we have supported by a direct evaluation of the evolution operator $Z_{\rm Reg}(R_2,R_1)$ defined by Eq. (14) of Ref. [3], representing the assumption by Diakonov and Petrov for Wilson loops in terms of the path integral over gauge degrees of freedom. As we have shown in Sec. VI the regularized evolution operator $Z_{\rm Reg}(R_2,R_1)$, evaluated correctly, is equal to zero. This agrees with our results obtained in Secs. IV and V. In Sec. VII we have shown that the removal of the oscillating factor from the evolution operator suggested in Ref. [3] via a shift of energy levels of the axial-symmetric top is prohibited. Such a shift of energy levels leads to a change of the starting symmetry of the system from $SU(2)$ to $U(2)$. By virtue of the oscillating factor the Wilson loop vanishes in the limit $I_\parallel, I_\perp\to 0$ in agreement with our results in Secs. IV, V, and VI.

We hope that the considerations in Secs. IV–VII are more than enough to persuade even the most distrustful reader that the path integral representation for the Wilson derived by Diakonov and Petrov by means of *special regularization and understanding* of the path integral over gauge degrees of freedom is erroneous.

The use of an erroneous path integral representation for Wilson loops in Ref. [7] has led to the conclusion that for large distances the average value of Wilson loops shows area–law falloff for any irreducible representation r of $SU(N)$. Unfortunately, this result is not supported by numerical simulations of lattice QCD [8]. At large distances, color charges with non-zero N-ality have string tensions of the corresponding fundamental representation, whereas color charges with zero N-ality are screened by gluons and cannot form a string. Therefore, the result obtained in Ref. [7] cannot be considered as *a new check of confinement in lattice calculations* as has been argued by the authors of Ref. [7].

We would like to accentuate that the problem we have touched in this paper is not of marginal interest and a path integral, if derived by means of an unjustified regularization procedure, would hardly compute the same physical number as the correct one. We argue that no regularization procedure can lead to specific dynamical constraints. In fact, the regularization procedure drawing the analogy with the axial-symmetric top has led to the result supporting the hypothesis of maximal Abelian projection pointed out by 't Hooft [14]. Any proof of this to full extent dynamical hypothesis through a regularization procedure and through specific *understanding* of the path integral should have seemed dubious and suspicious.

Finally, we have shown that within any path integral representation for Wilson loops in terms of gauge degrees of freedom no non-Abelian Stokes theorem in addition to Eq. (129) can be derived. Indeed, the Stokes theorem replaces a line integral over a closed contour by a surface integral with the closed contour as the boundary of a surface. However, approximating the path integral by an n-dimensional integral at $n\to\infty$ there are no closed paths linking two adjacent points along Wilson loops. Thereby, the line integrals over these open paths cannot be replaced by surface integrals. Thus, we argue that any non-Abelian Stokes theorem can be derived only within the definition of Wilson loops through the path ordering procedure (129). Of course, one can represent the surface-ordering operator $\mathcal{P}_S$ in Eq. (129) in terms of a path integral over gauge degrees of freedom, but this should not be a new non-Abelian Stokes theorem in comparison with the old one given by Eq. (129). That is why the claims of Refs. [3–5] concerning new versions of the non-Abelian Stokes theorems derived within path integral representations for Wilson loops seem unjustified.

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APPENDIX: COEFFICIENTS $a_j(z)$

In this appendix we evaluate the coefficients $a_j(z)$ of the expansion (51) and show that the completeness condition given by the series (59) converges to unity.

It is convenient to rewrite Eq. (51) as follows

$$\begin{aligned}e^{z\chi_{1/2}[t^3U]}&=\sum_j a_j(z)\,\chi_j[t^3U]\\&=a_0(z)+a_{1/2}(z)\,\chi_{1/2}[t^3U]+a_1(z)\,\chi_1[t^3U]\\&\quad+a_{3/2}(z)\,\chi_{3/2}[t^3U]+a_2(z)\,\chi_2[t^3U]\\&\quad+a_{5/2}(z)\,\chi_{5/2}[t^3U]+\cdots.\end{aligned} \tag{A1}$$

For the evaluation of the coefficients $a_0(z)$, $a_{1/2}(z)$, $a_1(z)$, $a_{3/2}(z)$, $a_2(z)$ and $a_{5/2}(z)$ we would make use the parametrization of the matrices U in terms of the Euler angles α, β and γ ranging over the regions $0\geqslant\alpha\geqslant 2\pi$, $0\geqslant\beta\geqslant\pi$ and $0\geqslant\gamma\geqslant 2\pi$, respectively [19]. Then, the Haar measure DU normalized to unity is defined by [19]

$$DU=\frac{1}{8\pi^2}\sin\beta\,d\beta\,d\alpha\,d\gamma. \tag{A2}$$

In the notation of Ref. [18] the characters are given by

$$\chi_{1/2}[t^3U]=-\,i\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2},$$

$$\chi_1[t^3U]=-2\,i\cos^2\frac{\beta}{2}\sin(\alpha+\gamma),$$

$$\chi_{3/2}[t^3U]=-12\,i\cos^3\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}\cos^2\frac{\alpha+\gamma}{2}+2\,i\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2},$$

$$\chi_2[t^3U]=-8\,i\cos^4\frac{\beta}{2}\sin(\alpha+\gamma)\cos(\alpha+\gamma)+i(1+\cos\beta)(1-2\cos\beta)\sin(\alpha+\gamma),$$

$$\chi_{5/2}[t^3U]=-80\,i\cos^5\frac{\beta}{2}\left(\sin\frac{\alpha+\gamma}{2}-2\sin^3\frac{\alpha+\gamma}{2}+\sin^5\frac{\alpha+\gamma}{2}\right)-i\left(15\cos^5\frac{\beta}{2}-12\cos^3\frac{\beta}{2}\right)\times\left(3\sin\frac{\alpha+\gamma}{2}-4\sin^3\frac{\alpha+\gamma}{2}\right)-i\left(10\cos^5\frac{\beta}{2}-12\cos^3\frac{\beta}{2}+3\cos\frac{\beta}{2}\right)\sin\frac{\alpha+\gamma}{2},\ \ldots \tag{A3}$$

The coefficients $a_0(z)$, $a_{1/2}(z)$, $a_1(z)$, $a_{3/2}(z)$, $a_2(z)$ and $a_{5/2}(z)$ are defined by the integrals (53) and (54):

$$a_0(z)=\int DU\,e^{z\,\chi_{1/2}[t^3U]}=\frac{1}{8\pi^2}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}d\beta\times\sin\beta\,e^{-i\,z\cos(\beta/2)\sin[(\alpha+\gamma)/2]},$$

$$a_{1/2}(z)=4\int DU\,\chi_{1/2}[t^3U^\dagger]\,e^{z\,\chi_{1/2}[t^3U]}=\frac{i}{2\pi^2}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}d\beta\times\sin\beta\left[\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}\right]e^{-i\,z\cos(\beta/2)\sin[(\alpha+\gamma)/2]},$$

$$a_1(z)=\frac{3}{2}\int DU\,\chi_1[t^3U^\dagger]\,e^{z\,\chi_{1/2}[t^3U]}=\frac{3i}{8\pi^2}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}d\beta\sin\beta\times\left[\cos^2\frac{\beta}{2}\sin(\alpha+\gamma)\right]e^{-i\,z\cos(\beta/2)\sin[(\alpha+\gamma)/2]},$$

$$a_{3/2}(z)=\frac{4}{5}\int DU\,\chi_{3/2}[t^3U^\dagger]\,e^{z\,\chi_{1/2}[t^3U]}=\frac{i}{5\pi^2}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}d\beta\sin\beta\times\left[6\cos^3\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}\cos^2\frac{\alpha+\gamma}{2}-\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}\right]e^{-i\,z\cos(\beta/2)\sin[(\alpha+\gamma)/2]},$$

$$a_2(z)=\frac{1}{2}\int DU\,\chi_2[t^3U^\dagger]\,e^{z\,\chi_{1/2}[t^3U]}=\frac{i}{16\pi^2}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}d\beta\sin\beta\times\left[8\cos^4\frac{\beta}{2}\sin(\alpha+\gamma)\cos(\alpha+\gamma)-(1+\cos\beta)(1-2\cos\beta)\sin(\alpha+\gamma)\right]\times e^{-i\,z\cos(\beta/2)\sin[(\alpha+\gamma)/2]},$$

$$a_{5/2}(z)=\frac{12}{35}\int DU\,\chi_{3/2}[t^3U^\dagger]\,e^{z\,\chi_{1/2}[t^3U]}=\frac{3i}{70\pi^2}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}d\beta\sin\beta\times\left[80\cos^5\frac{\beta}{2}\left(\sin\frac{\alpha+\gamma}{2}-2\sin^3\frac{\alpha+\gamma}{2}+\sin^5\frac{\alpha+\gamma}{2}\right)+\left(15\cos^5\frac{\beta}{2}-12\cos^3\frac{\beta}{2}\right)\times\left(3\sin\frac{\alpha+\gamma}{2}-4\sin^3\frac{\alpha+\gamma}{2}\right)+\left(10\cos^5\frac{\beta}{2}-12\cos^3\frac{\beta}{2}+3\cos\frac{\beta}{2}\right)\times\sin\frac{\alpha+\gamma}{2}\right]e^{-i\,z\cos(\beta/2)\sin[(\alpha+\gamma)/2]},\quad\ldots. \tag{A4}$$

The integration over angle variables gives

$$a_0(z)=2\,\frac{J_1(z)}{z},\quad a_{1/2}(z)=8\,\frac{J_2(z)}{z},\quad a_1(z)=0,$$
$$a_{3/2}(z)=\frac{16}{5}\,\frac{J_4(z)}{z},\quad a_2(z)=0,$$
$$a_{5/2}(z)=\frac{72}{35}\,\frac{J_6(z)}{z},\quad\ldots. \tag{A5}$$

One can show that for an arbitrary j the coefficient $a_j(z)$ is defined by

$$a_j(z)=\begin{cases}\dfrac{3(2j+1)}{j(j+1)}\dfrac{J_{2j+1}(z)}{z}, & j=1/2,3/2,5/2,\dots,\\ 0, & j=1,2,\dots .\end{cases} \tag{A6}$$

This implies the following general formula:

$$\int DU\,\chi_j[t^3U^\dagger]\,e^{z\,\chi_{1/2}[t^3U]} =\begin{cases}(2j+1)\dfrac{J_{2j+1}(z)}{z}, & j=1/2,3/2,5/2,\dots,\\ 0, & j=1,2,\dots .\end{cases} \tag{A7}$$

The completeness condition (59) for arbitrary z takes the form

$$4\,\frac{J_1^2(z)}{z^2}+3\sum_{j=1/2}^{\infty}\frac{(2j+1)^2}{j(j+1)}\frac{J_{2j+1}^2(z)}{z^2}=1. \tag{A8}$$

The numerical values of $a_0(z)$, $a_{1/2}(z)$, $a_{3/2}(z)$ and $a_{5/2}(z)$ for the particular case $z=1$ read

$$a_0(1)=0.88,\quad a_{1/2}(1)=0.92,\quad a_{3/2}(1)=0.01,\quad a_{5/2}(1)=4\times10^{-5}. \tag{A9}$$

For the completeness condition (59) we obtain at $z=1$

$$a_0^2(1)+\sum_{j=1/2}^{\infty}\frac{1}{3}j(j+1)\,a_j^2(1) =a_0^2(1)+\frac{1}{4}a_{1/2}^2(1)+\frac{5}{4}a_{3/2}^2(1)+\frac{35}{12}a_{5/2}^2(1)+\cdots=0.774 +0.212+1.25\times10^{-4}+5\times10^{-9}+\cdots =0.986+\cdots. \tag{A10}$$

Thus, this series converges slowly to unity.

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